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USE OF SYMBOLIC MANIPULATION TECHNIQUES TO EXAMINE GRAVITATIONAL AND UNIFIED FIELD THEORIES

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Perception Technology Corporation

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This is the final technical report for the contract entitled "Use of Symbolic Manipulation Techniques to Examine Gravitational and Unified Field Theories". The report consists of four sections. The first three sections deal with the three contractual committments as outlined in Item No. 0001. Scope of Work. Section 5 contains seven appendices dealing with additional, related studies we have made during the year June 30, 1973 to June 15, 1974.

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USE OF SYMBOLIC MANIPULATION TECHNIQUES TO EXAMINE GRAVITATIONAL AND UNIFIED FIELD THEORIES

FINAL TECHNICAL REPORT

Contract No. DAHC 15 73 C 0369

Sponsored by

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1400 Wilson Boulevard
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USE OF SYMBOLIC MANIPULATION TECHNIQUES TO EXAMINE GRAVITATIONAL AND UNIFIED FIELD THEORIES

FINAL TECHNICAL REPORT

0. INTRODUCTION

This is the final technical report for work performed by Perception Technology Corporation under Contract No. DAH15 73 C 0369. Three main objectives of the work defined as "Scope of Work" in the contract, Item No. 0001 have been satisfactorily completed. These three items are:

(1) Verification of the conjecture that a functional expansion (equations 7-9 of the proposal) satisfies, up to a divergence, the field equations (11 of the proposal). Using MACSYMA we have proven that the conjecture is indeed correct. The proof is general and covers the full time-dependent case for the first and second order expansions of the metric. A detailed description of this verification is given in Section 1.

(2) Verification of the conjecture that, when conservation conditions $\partial_\nu(\sqrt{-g}\,\sigma u_\mu u^\nu) = 0$ are imposed, the vanishing covariant divergence of (11) reduces to the geodesic equations of motion (12). Using MACSYMA we have proven that this conjecture is also correct.

The computation covers the full time-dependent case and includes first and second order expansions as promised. Details of the computation are given in Section 2.

(3) Study of a Unified Field Theory approach described in the contract is carried into first and second order expansions of the functional metric. The theory works out satisfactorily in the first order approximation. In the second order the stress-energy tensor is found to differ from Maxwell's tensor by an extra term. We have not been able to remove this term, nor do we understand fully its significance. Because of this term the trace of the stress-energy tensor does not vanish. Hence, the Lagrangian does not reduce to its special relativistic limit of correspondence. However, we are not abandoning the unified theory as unworkable. Already it comes close to being satisfactory so that efforts toward a reasonable re-interpretation is continuing. The details of the computations appear in Section 3.

(4) In the course of the above calculations the capabilities of MACSYMA had to be extended by our company in several respects. These extensions are all important from the point of view of MACSYMA as a general mathematical tool and are described in Section 4.

(5) Several physical problems were investigated as additional items not listed in the contract. These appear in Appendices I-VII and have to do mainly with an overall comparison of the new theory vis a vis the conventional theory of Einstein. The final Appendix represents a concise formulation of the new theory of gravitation as we understand it at the present time.

1. VERIFICATION OF THE FIELD EQUATIONS

The field equations are given by

$$R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R = 2(\Box^2\phi_\mu^\nu + t_\mu^\nu) + \partial_\alpha \varepsilon_\mu^{\nu\alpha} \tag{1.1}$$

where the left hand side is the Einstein tensor, the expression $\Box^2\phi_\mu^\nu \equiv g^{\alpha\beta}\phi^\nu_{\mu|\alpha\beta}$ is the general tensor d'Alembertian. The slash denotes covariant differentiation and

$$t_\mu^\nu = -2(\partial_\mu\phi^\alpha_\beta\partial^\nu\phi^\beta_\alpha - \frac{1}{2}\,\delta_\mu^\nu\partial^\lambda\phi^\alpha_\beta\partial_\lambda\phi^\beta_\alpha) + \partial_\mu\phi\partial^\nu\phi - \frac{1}{2}\,\delta_\mu^\nu\partial^\lambda\phi\partial_\lambda\phi \tag{1.2}$$

is the stress energy tensor of the gravitational field. The quantity $\partial_\alpha\varepsilon_\mu^{\nu\alpha}$ is a particular ordinary divergence whose significance is described below. In the theory the components of $g_{\mu\nu}$ are functions of the field tensor and satisfy, to second order, the relation

$$g_{\mu\nu} = \eta_{\mu\nu}(1 + 2\phi + 2\phi^2) - 4\phi_{\mu\nu} - 8\phi\phi_{\mu\nu} + 8\phi^\sigma_\mu\phi_{\nu\sigma} \tag{1.3}$$

Here ϕ is the trace of ϕ^μ_ν and $\eta_{\mu\nu}$ is the Lorentz matrix. A general relation, of which (1.3) is the second order expansion, is given in Appendix VII. An additional equation in the theory is

$$\partial_\mu\phi^\mu_\nu = 0 \qquad \text{or} \qquad \eta^{\nu\sigma}\partial_\sigma\phi^\mu_\nu = \partial^\nu\phi^\mu_\nu = 0 \tag{1.4}$$

This is the "Lorentz condition" which implies the local isotropic constancy of the velocity of light. This physical condition is not present in the Einstein theory except for the linearized (first order) Einstein solution where it appears as a coordinate condition.

Our purpose, using MACSYMA was to prove that substitution of (1.3) into the left hand side of (1.1) results, to second order, in the right hand side of (1.1).

The definition of the Ricci tensor in terms of the metric tensor is given by

$$R^{\mu}_{\nu} = g^{\mu\sigma} \left[\partial_{\alpha}\{^{\alpha}_{\nu\sigma}\} - \partial_{\sigma}\{^{\alpha}_{\nu\alpha}\} + \{^{\beta}_{\nu\sigma}\}\{^{\alpha}_{\beta\alpha}\} - \{^{\beta}_{\nu\alpha}\}\{^{\alpha}_{\beta\sigma}\}\right] \tag{1.5}$$

where the Christoffel symbol $\{^{\alpha}_{\mu\sigma}\} \equiv \frac{1}{2} g^{\beta\alpha}(\partial_{\sigma}g_{\mu\beta} + \partial_{\mu}g_{\sigma\beta} - \partial_{\beta}g_{\mu\sigma})$.

The calculation proceeds in the following manner: The relation (1.3) is rewritten as

$$g_{\mu\nu} = \eta_{\mu\nu} + \lambda(2\phi\eta_{\mu\nu} - 4\phi_{\mu\nu}) + \lambda^{2}(2\phi^{2}\eta_{\mu\nu} - 8\phi\phi_{\mu\nu} + 8\phi^{\sigma}_{\mu}\phi_{\sigma\nu}) \tag{1.6}$$

where λ is an expansion parameter. To compute Christoffel symbols one first needs the contravariant metric tensor $g^{\mu\nu}$ which is easily found from (1.6) via the relation

$$g_{\mu\nu}g^{\nu\sigma} = \delta^{\sigma}_{\mu} + O(\lambda^{3}) \tag{1.7}$$

One finds

$$g^{\mu\nu} = \eta^{\mu\nu} - \lambda(2\phi\eta^{\mu\nu} - 4\phi^{\mu\nu}) + \lambda^2(2\phi^2\eta^{\mu\nu} - 8\phi\phi^{\mu\nu} + 8\phi^{\mu\sigma}\phi^{\nu}_{\sigma}) \qquad (1.8)$$

The Christoffel symbols as well as the Ricci tensor may then be computed in a straightforward manner. Substitution of (1.6) and (1.8) into (1.5) results in an expression which contain terms of order higher than the second. Accordingly, upon expansion of the Ricci tensor, all terms containing $\lambda^{\eta} (\eta > 2)$ are dropped. The resulting second order expansion for R^{ν}_{μ} is found to have 144 terms. This number would have been literally thousands if MACSYMA did not have indicial manipulation capability.

One of the reasons for the relatively large number of terms at this stage is the implementation of a program to avoid the use of dummy indices more than twice. Thus, each time a dummy index occurs twice a counter names the next pair of indices with a larger value. Once the final expression is obtained a function called rename cause dummy indices in each element of the expression to be renamed beginning with the counter 1. Thus many of the terms cancel in the simplifier without the implementation of any symmetry properties. We find rename(R^{ν}_{μ}) reduces the expression to 124 terms. The next step is the implementation of the Lorentz condition (1.4). Many terms vanish as a consequence of this relation and we are left with 50. The final step is the contraction of the terms with respect to the Lorentz matrix. We are then left

with 31 terms in the expression o[illegible]. We form the scalar curvature $R = R^{\mu}_{\mu}$ and apply the functions Lorentz, contract and rename once more. We are left with 27 terms in R. The resulting expression for $R^{\mu}_{\nu} - \frac{1}{2}\,\delta^{\mu}_{\nu}R \equiv G^{\mu}_{\nu}$ contains 56 terms as not all terms are unique. This expression of the Einstein tensor is further manipulated and reduced to

$$
\begin{aligned}
G^{\nu}_{\mu} = {} & \delta^{\nu}_{\mu}(2\partial^{\lambda}\phi^{\alpha}_{\beta}\partial_{\lambda}\phi^{\beta}_{\alpha} - \partial^{\lambda}\phi\partial_{\lambda}\phi) + 2\partial_{\mu}\phi\partial^{\nu}\phi - 4\partial_{\mu}\phi^{\alpha}_{\beta}\partial^{\nu}\phi^{\beta}_{\alpha} \\
& + 2\Box^{2}_{0}\phi^{\nu}_{\mu} + 4\phi^{\nu}_{\alpha}\Box^{2}_{0}\phi^{\alpha}_{\mu} - 4\phi^{\alpha}_{\mu}\Box^{2}_{0}\phi^{\nu}_{\alpha} - 4\phi\,\Box^{2}_{0}\phi^{\nu}_{\mu} - 4\phi^{\alpha}_{\beta}\partial_{\alpha}\partial^{\nu}\phi^{\beta}_{\mu} \\
& + 4\partial_{\beta}\phi^{\alpha}_{\mu}\partial^{\nu}\phi^{\beta}_{\alpha} + 8\phi^{\alpha}_{\beta}\partial_{\alpha}\partial^{\beta}\phi^{\nu}_{\mu} - 8\partial_{\beta}\phi^{\alpha}_{\mu}\partial^{\beta}\phi^{\nu}_{\alpha} - 4\phi^{\beta}_{\alpha}\partial_{\mu}\partial_{\beta}\phi^{\alpha\nu} \\
& + 4\partial_{\mu}\phi^{\beta}_{\alpha}\partial_{\beta}\phi^{\alpha\nu}
\end{aligned}
\tag{1.9}
$$

where $\Box^{2}_{0}$ is the special relativistic d'Alembertian defined by

$$\Box^{2}_{0} \equiv \eta^{\alpha\beta}\,\frac{\partial}{\partial x^{\alpha}}\,\frac{\partial}{\partial x^{\beta}} \tag{1.10}$$

At this stage it is convenient to check that (1.8) is correct. This may be accomplished by computing the covariant divergence of (1.8). By definition

$$G^{\mu}_{\nu|\mu} \equiv \partial_{\mu}G^{\mu}_{\nu} - \left\{{\sigma \atop \alpha\sigma}\right\} G^{\alpha}_{\nu} + \left\{{\alpha \atop \nu\sigma}\right\} G^{\sigma}_{\alpha} \tag{1.11}$$

and it is well known that the Einstein tensor, for any metric, has a vanishing tensor divergence. Using MACSYMA we have explicitly proven that divergence is indeed zero.

If we now examine (1.9) in detail it is seen that the sum of the first, second, third and fourth terms is $2t_\mu^\nu$ as desired. It remains to prove that the remaining terms in (1.9) are equal, within an ordinary divergence, to the general tensor d'Alembertian of ϕ_μ^ν for the metric (1.3). From the definition (1.1) it is found

$$\Box^2\phi_\mu^\nu = \Box_0^2\phi_\mu^\nu + 4\phi_\beta^\alpha\partial_\alpha\partial^\beta\phi_\mu^\nu - 2\phi\Box_0^2\phi_\mu^\nu - 8\partial_\beta\phi_{\alpha\mu}\partial^\alpha\phi^{\nu\beta}$$
$$- 2\phi_\mu^\alpha\Box_0^2\phi_\alpha^\nu + 2\partial^\alpha\phi\partial_\mu\phi_\alpha^\nu + 2\phi_\alpha^\nu\Box_0^2\phi_\mu^\alpha + 4\partial_\beta\phi_{\mu\alpha}\partial^\nu\phi_\beta^\alpha$$
$$+ 2\partial^\alpha\phi\,\partial^\nu\phi_{\mu\alpha} + 4\partial_\mu\phi_{\alpha\beta}\partial^\beta\phi^{\alpha\nu} \qquad (1.12)$$

Substituting (1.12) and (1.2) into (1.9) it is found

$$G_\mu^\nu - 2\Box^2\phi_\mu^\nu - 2t_\mu^\nu = - 4\phi_\beta^\alpha\partial^\beta\partial^\nu\phi_{\alpha\mu} + 8\partial_\beta\phi_{\mu\alpha}\partial^\alpha\phi^{\nu\beta} - 4\partial_\mu\phi_{\alpha\beta}\partial^\beta\phi^{\alpha\nu}$$
$$- 4\partial_\beta\phi_{\mu\alpha}\partial^\nu\phi^{\alpha\beta} - 4\phi_\beta^\alpha\partial^\beta\partial_\mu\phi_\alpha^\nu - 4\partial^\alpha\phi\partial_\mu\phi_\alpha^\nu - 4\partial^\alpha\phi\,\partial^\nu\phi_{\alpha\mu} \qquad (1.13)$$

Employing the Lorentz condition (1.4) it is easily seen that the right hand side of (1.13) may be written as a divergence, $\partial_\alpha\varepsilon_\mu^{\nu\alpha}$, where

$$\varepsilon_{\mu}^{\nu\alpha} = -4\phi^{\beta\alpha}\partial^{\nu}\phi_{\beta\mu} + 8\phi_{\mu\beta}\partial^{\beta}\phi^{\nu\alpha} - 4\phi^{\beta\nu}\partial_{\mu}\phi_{\beta}^{\alpha} - 4\phi_{\mu\beta}\partial^{\nu}\phi^{\beta\alpha}$$

$$-4\phi^{\beta\alpha}\partial_{\mu}\phi_{\beta}^{\nu} - 4\phi\partial_{\mu}\phi^{\nu\alpha} - 4\phi\partial^{\nu}\phi_{\mu}^{\alpha} \tag{1.14}$$

Hence, we have established (1.1) and our conjecture is verified. We may emphasize again that this computation would be literally prohibitive if carried out by hand in its full generality.

In concluding this section it will be worthwhile to point out that in the quasi-static limit where the heavy particles generating the field are slow the last term of (1.1) vanishes on account of the Lorentz condition, hence, the equations exhibit an exact correspondence with a special relativistic tensor field theory.

2. EQUATIONS OF MOTION AND CONSERVATION LAWS

We have seen that to second order

$$R^{\nu}_{\mu} - \frac{1}{2}\,\delta^{\nu}_{\mu}R = 2(\Box^2\phi^{\nu}_{\mu} + t^{\nu}_{\mu}) + \partial_{\sigma}\varepsilon^{\nu\sigma}_{\mu} \tag{2.1}$$

We shall now discuss our investigations on the Scope of the Work, Part 2 of the contract. In this phase we examine the vanishing of the tensor divergence of (2.1). Since the covariant divergence of the left hand side of (2.2) vanishes identically it is clear that the divergence of the right hand side also vanishes. We wish to discover whether this vanishing, together with the conservation laws of the theory, lead to the geodesic equations of motion. Thus, we shall examine (to second order)

$$[2(\Box^2\phi^{\nu}_{\mu} + t^{\nu}_{\mu}) + \varepsilon^{\nu\alpha}_{\mu,\alpha}]_{|\nu} = 0 \tag{2.2}$$

subject to

$$\partial_{\nu}(\sqrt{g}\;\sigma u_{\mu}u^{\nu}) = 0 \tag{2.3}$$

From Appendix VII, equation (2.9) we have

$$t^{\nu}_{\mu} = -2(\partial_{\mu}\phi^{\alpha}_{\beta}\;\partial^{\nu}\phi^{\beta}_{\alpha} - \frac{1}{2}\,\delta^{\nu}_{\mu}\partial^{\lambda}\phi^{\alpha}_{\beta}\;\partial_{\lambda}\phi^{\beta}_{\alpha}) + \partial_{\mu}\phi\;\partial^{\nu}\phi - \frac{1}{2}\,\partial^{\lambda}\phi\partial_{\lambda}\phi \tag{2.4}$$

Since this is a second order quantity it follows that

$$t_{\mu|\nu}^{\nu} = \partial_\nu t_\mu^\nu = \partial_\mu \phi \Box_0 \phi - 2\partial_\mu \phi_{\alpha\beta} \Box_0 \phi^{\alpha\beta} \qquad (2.5)$$

For any second rank tensor S_ν^μ we have

$$S_{\mu|\nu}^{\nu} \equiv \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g}\, S_\mu^\nu) - \{{}_{\mu\beta}^{\alpha}\}\, S_\alpha^\beta \qquad (2.6)$$

as an identity. From Appendix VII, equation (1.1)

$$\Box^2 \phi_\mu^\nu = 4\pi\sigma u_\mu u^\nu \qquad (2.7)$$

From (2.6) and (2.7) we thus have

$$\frac{1}{4\pi} (\Box^2 \phi_\mu^\nu)|_\nu \equiv \frac{1}{\sqrt{g}} \partial_\beta (\sqrt{g}\, \sigma u_\mu u^\beta) - \sigma \{{}_{\mu\beta}^{\alpha}\}\, u^\beta u_\alpha \,. \qquad (2.8)$$

However, from the definition of the metric we have

$$\sigma\{{}_{\mu\beta}^{\alpha}\}\, u_\alpha u^\beta = \sigma \partial_\mu \phi - 2\sigma u^\alpha u^\beta\, \partial_\mu \phi_{\alpha\beta} \qquad (2.9)$$

and from (2.7)

$$\Box^2 \phi = 4\pi\sigma \qquad (2.10)$$

Therefore from (2.7) and (2.10), (2.8) becomes

$$(\Box^2\phi_\mu^\beta)|_\beta = \frac{4\pi}{\sqrt{g}}\,\partial_\beta(\sqrt{g}\;\sigma u_\mu u^\beta) - \partial_\mu\phi\,\Box_0\phi + 2\partial_\mu\phi_{\alpha\beta}\Box_0\phi^{\alpha\beta} \tag{2.11}$$

Thus, from (2.5) and (2.11) we find (2.2) becomes

$$8\pi(\sqrt{-g})^{-1}\;\partial_\nu(\sqrt{g}\;\sigma u_\mu u^\nu) + \partial_\alpha\partial_\beta\varepsilon_\mu^{\alpha\beta} = 0 \tag{2.12}$$

since $\varepsilon_\mu^{\alpha\beta}$ is also a second order quantity. This is the conservation law for the new theory. In the quasi-static limit $\partial_\alpha\partial_\beta\varepsilon_\mu^{\alpha\beta} = O(\phi^3)$ and we are left with $\partial_\nu(\sqrt{g}\;\sigma u_\mu u^\nu) = 0$. We believe that for non-static metrics $\partial_\alpha\partial_\beta\varepsilon_\mu^{\alpha\beta}$ is involved in gravitational radiation. This subject will be investigated in the next phase of the contract.

Let us now return to (2.2) which, from (2.8) yields (to second order)

$$8\pi\;\left(\frac{1}{\sqrt{g}}\;\partial_\beta(\sqrt{g}\;\sigma u_\mu u^\beta) - \sigma\;\{^{\alpha}_{\mu\beta}\}\;u^\beta u_\alpha\right) + 2\partial_\beta t_\mu^\beta + \partial_\alpha\partial_\beta\varepsilon_\mu^{\alpha\beta} \tag{2.13}$$

From (2.12) this reduces to

$$-\;\sigma\;\{^{\alpha}_{\mu\beta}\}\;u^\beta u_\alpha + \partial_\beta\;(t_\mu^\beta/4\pi) = 0 \tag{2.14}$$

However, from correspondence with special relativity and other local field theories the equations of motion of test particles are given by

$$\sigma\;\frac{d^2x_\mu}{ds^2} = \partial_\nu\;(t_\mu^\nu/4\pi) \tag{2.15}$$

Hence, from (2.14) and (2.15) we find the geodesic equations of motion

$$\frac{d^2x_\mu}{ds^2} = \{{}^{\alpha}_{\mu\beta}\}\, u^\beta u_\alpha \tag{2.16}$$

which are, of course, equivalent to their more familiar form

$$\frac{d^2x^\mu}{ds^2} + \{{}^{\mu}_{\alpha\beta}\}\, u^\alpha u^\beta = 0 \tag{2.17}$$

Thus, we have verified that the geodesic equations of motion (2.17) follow, up to a divergence, from the field equations (2.2) upon application of the conservation laws $\partial_\nu(\sqrt{-g}\,\sigma u_\mu u^\nu) = 0$. We have thus verified the conjecture that in first and second order the geodesic equations of motion are a consequence of the field equations plus the conservation laws. This statement is rigorously valid in the quasistatic limit. In the most general fully time-dependent case the divergence terms seem to be related to the radiation reaction and radiative damping. In the quasi-static limit radiation tends to zero, divergence terms vanish, hence the geodesic equations of motion become rigorously valid.

3. SKEW-SYMMETRIC $\phi_{\mu\nu}$ AND UNIFIED FIELD THEORY

Our methods in the third phase of the contract do not follow the standard approach to unified field theories. Our philosophy is as follows: We know that to second order (Appendix 7, (2.6)), the definition

$$\tilde{g} = \tilde{\eta} \cdot e^{2(\phi - 2\tilde{\phi})} \tag{3.1}$$

produces a viable theory of gravitation when the Einstein tensor $G_{\mu\nu}$ is evaluated for symmetric $\phi_{\lambda\nu}$. The obvious question to be considered concerns the form $G_{\mu\nu}$ takes when $\phi_{\lambda\nu}$ is skew-symmetric. Skew-symmetry of $\phi_{\lambda\nu}$ implies $\phi = 0$, hence we consider the metric tensor to take the form (to second order)

$$g_{\mu\nu} = \eta_{\mu\nu} + \lambda\varepsilon_{\mu\nu} + \frac{1}{2}\lambda^2 \cdot \alpha\varepsilon_{\mu\cdot}^{\;\rho}\,\varepsilon_{\rho\nu} + \ldots \tag{3.2}$$

where $\eta_{\mu\nu}$ is the symmetric Lorentz matrix but now $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$. The parameter α in (3.2) takes the value ± 1 and can only be determined later via a correspondence principle. λ is an expansion parameter such that $\lambda << 1$. We do not require the second order contravariant metric tensor but will need the first order part. It is found that

$$g^{\mu\nu} = \eta^{\mu\nu} - \lambda\varepsilon^{\mu\nu} \tag{3.3}$$

satisfies $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu} + O(\lambda^2)$ and hence (3.3) is the relation we adopt.

The only postulate we shall employ is similar to other unified theories. Namely, the metric tensor is to have a vanishing covariant derivative. Thus

$$g_{\mu\nu;\sigma} \equiv \partial_\sigma g_{\mu\nu} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} - g_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu} \equiv 0 \tag{3.4}$$

From this relation we may now obtain the connection coefficients $\Gamma^{\alpha}_{\mu\sigma}$ for the metric (3.2) which we write in the more convenient form

$$g_{\mu\nu} = \eta_{\mu\nu} + \lambda\varepsilon_{\mu\nu} + \lambda^2 h_{\mu\nu} \quad . \tag{3.5}$$

Since, in metrical theories, connection coefficients are linear in first derivatives of the metric and since $\eta_{\mu\nu}$ is constant with respect to coordinate differentiation it follows that to first order (3.4) and (3.5) imply

$$\partial_\sigma\varepsilon_{\mu\nu} - \eta_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} - \eta_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu} = 0. \tag{3.6}$$

The solution of this relation is

$$\Gamma^{\alpha}_{\mu\sigma} = \eta^{\rho\alpha}\ (\mu\rho,\sigma)_{\varepsilon} \tag{3.7}$$

where we adopt the notation

$$(\mu\rho,\sigma)_\varepsilon \equiv \frac{1}{2}(\partial_\sigma \varepsilon_{\mu\rho} + \partial_\mu \varepsilon_{\rho\sigma} - \partial_\rho \varepsilon_{\sigma\mu}) \qquad \textbf{(3.8)}$$

Using an identical method, with the notation in (3.8) we find the second order expression for $\Gamma^\alpha_{\mu\sigma}$ to be

$$\Gamma^\alpha_{\mu\sigma} = \eta^{\rho\alpha}\,[(\mu\rho,\sigma)_\varepsilon + (\mu\rho,\sigma)_\psi] \qquad (3.9)$$

where

$$\partial_\sigma \psi_{\mu\rho} \equiv \partial_\sigma h_{\mu\rho} - \varepsilon^{\theta}_{\boldsymbol{\cdot}\rho}(\mu\theta,\sigma)_\varepsilon - \varepsilon_{\mu\boldsymbol{\cdot}}^{\ \ \theta}(\sigma\theta,\rho)_\varepsilon \ . \qquad (3.10)$$

Substitution of (3.10) into (3.9) results in 25 terms for $\Gamma^\alpha_{\mu\sigma}$. This expression is all we need to examine geometrical tensors in this framework. This is as far as our calculation proceeded by hand. MACSYMA was then given the definition (3.9) and (3.10).

We first note that the first set of Maxwell's equations can be written as

$$\partial_\nu(\sqrt{-g}\ g^{\mu\nu}) = 4\pi J^\nu \qquad (3.11)$$

Then the second set is obtainable as

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\alpha\beta} = 0 \qquad (3.12)$$

From (3.8) one can further prove the Lorentz force

$$\sigma \frac{d^2x_\mu}{ds^2} = \sigma \left\{ {\alpha \atop \mu\beta} \right\} u_\alpha u^\beta = F_{\mu\alpha} J^\alpha \tag{3.13}$$

One can therefore see that the first order equations of electrodynamics can be accommodated. The problem then reduces to a second order calculation to see if the Maxwell tensor is also obtainable.

Given $\Gamma^\alpha_{\mu\sigma}$ to second order we shall construct the Ricci tensor using MACSYMA, via the definition

$$R_{ik} \equiv \Gamma^s_{ik,s} - \Gamma^s_{is,k} + \Gamma^s_{ts}\Gamma^t_{ik} - \Gamma^s_{tk}\Gamma^t_{is} \tag{3.14}$$

In the standard version of Tensor Fasl (Section 4), (3.14) may be expanded to second order with no difficulty as MACSYMA does not sort indices. The expression for R_{ik} contains 102 terms which is reduced to 90 using rename. A new version of Tensor Fasl was written to enable the contraction of expressions when the order of indices is important as in this case. Contracting R_{ik}, employing the Lorentz condition $\varepsilon^{\mu\nu}{}_{,\nu} = 0$ and grouping all divergence we find that

$$R_{\mu\nu} = \frac{1}{4} [\partial_\mu \varepsilon_{\alpha\beta}\, \partial_\nu \varepsilon^{\alpha\beta} - 2\partial_\beta \varepsilon_{\mu\alpha}\, \partial^\beta \varepsilon_{\nu\cdot}{}^{\alpha} - 2\,\Box^2_0 \varepsilon_{\mu\nu}] + \text{Div.} \tag{3.15}$$

where $\Box^2_0 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. This result has been obtained by requiring that the expression for $R_{\mu\nu}$ reduce to the symmetric case ($\phi = 0$) when

$\varepsilon_{\mu\nu} = 4\phi_{\mu\nu} = \varepsilon_{\nu\mu}$. This is the correspondence limit which gives $\alpha = 1$ in (3.2).

From (3.12) and (3.3) it follows that

$$R = (\eta^{\mu\nu} - \lambda\varepsilon^{\mu\nu})R_{\mu\nu} = -\frac{1}{4}\lambda^2\varepsilon_{\mu\nu,\sigma}\varepsilon^{\mu\nu,\sigma} \tag{3.16}$$

From (3.12) and (3.13) we find

$$G_{\mu\nu} = \frac{1}{4}[\partial_\mu\varepsilon_{\alpha\beta}\partial_\nu\varepsilon^{\alpha\beta} - 2\partial_\beta\varepsilon_{\mu\alpha}\partial^\beta\varepsilon_{\nu\cdot}^{\ \alpha} - 2\Box_0^2\varepsilon_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\partial_\sigma\varepsilon_{\alpha\beta}\partial^\sigma\varepsilon^{\alpha\beta}] + \text{Div.} \tag{3.17}$$

Note that this is precisely the same result we obtain in the symmetric case if we set $\phi = 0$ and choose $\varepsilon_{\mu\nu} = -4\phi_{\nu\mu}$.

Note that the replacement of $\Box^2$ instead of $\Box_0^2$ would only bring antisymmetric terms hence would not alter the essential character of the result. Two of the three non-linear terms in (3.14) are exactly of the form of a Maxwell stress-tensor. The only term which is troublesome is the first one. At the present time we do not know what to do with this result except to say that the whole scheme almost works out but doesn't quite make it because of the extra term. Future investigations may indicate a way to overcome this problem or help us to clarify the situation.

In conclusion although the unified field theory direction did not yet work out to our satisfaction it showed sufficient promise that it may do so by a suitable reinterpretation. In any case it will be fair to say that with the help of the MACSYMA we were able to carry the subject further than any other calculation made in the past.

4. NEW CAPABILITIES OF MACSYMA
THE TENSOR PACKAGE

The MACSYMA system consists of over 250 functions for the manipulation of mathematical expressions. There are not only functions for performing mathematical computations but also those for many necessary miscellaneous manipulations such as extracting parts, simplifying, input output, etc. Each one of these if made up of one or as many as several hundred programs (in the case of integration) written in the LISP language. The expressions dealt with are composed of operators and arguments to them which are numbers, variables, functions references, arrays, lists, and matrices. The mathematical knowledge built for MACSYMA is continually expanding as more and more programs are written to provide additional capabilities. At present there are over 3000.

If one wishes to introduce a new mathematical entity into the system then he would not only have to cast it in terms of the existing syntax but also it would be necessary to provide the programs to perform the desired manipulations on it as it would be foreign to most of the existing programs. Perception Technology Corporation has embarked on an effort to create facility for dealing with tensor manipulation as MACSYMA had nothing that would provide a convenient,

efficient, and workable mechanism for indicial types of manipulations. The basic program we have created is called "Tensor Fasl". This program is now quite sophisticated and we would estimate it contains 40% of the tensor analysis found in standard texts. However, it is capable of solving problems in tensor calculus that are impractical to do by hand.

Since tensors are defined not by the way they are written but by certain properties they have, we represent them as more general quantities which we call indexed objects. They possess a name and two lists of indices, the first list being the subscripts (covariant indices) and the second list being the superscripts (contravariant indices). These indexed objects are represented as functions of two arguments, which are the above mentioned lists. If either set of indices is absent it is represented by an empty list. Thus δ_i^j and A^{ijk} would be represented as δ([i],[j]) and A([ ],[i,j,k]).

Ordinary differentiation of an indexed object (say E) with respect to the coordinate x^k is obtained by the command DIFF(E,k). Since MACSYMA wouldn't know that E depends on x^k it would give 0 but the differentiation program in Tensor Fasl has been modified to assume that all indexed objects depend on all coordinates. If an indexed object is independent of all coordinates this is stated by the command DECLARE(E,constant) and subsequent differentiation of E would yield 0. The derivative of an indexed object causes the coordinate with respect to which differentiation is carried out to

be appended to the function which represents the indexed object in MACSYMA. The indices which denote differentiation with respect to a coordinate are sorted in alphabetic order to take advantage of the commutativity of differentiation in order to simplify expressions composed of those indexed objects.

At present three basic features have been implemented - symmetries and contractions, and covariant differentiation. For the first three functions are provided. One for declaring symmetries (e.g. G([ ],[j,i]) is equal to G([ ],[i,j])), one for removing them, and one for displaying them in case the user forgot which ones he declared. For the purpose of contracting products of indexed objects in expressions, four functions were provided. One to declare that some indexed object contracts with another to form a third, another function to remove these declarations, another to display them, and also a function to perform the contractions. This latter function is rather difficult to code as it is necessary to make several passes over the expression to make sure that all possible contractions have been utilized. The function also performs the substitutions implied by the Kronecker delta function if it occurs in an expression. In addition a function 'Show' has been written to display expressions containing indexed objects in more natural notation than MACSYMA ordinarily would. MACSYMA assumes that A([i,j],[k],l,m) is a function of four arguments, and would display it as such, whereas we like to see it displayed

as $A^{k}_{ij,lm}$. Tensor Fasl also performs multiple covariant differentiation on indexed objects with any number of indices. These expressions may be expanded in terms of Christoffel symbols of the second kind (and their derivatives). In addition, numerical tensors are defined as well as the relation between connection and metric.

The following list of capabilities of the MACSYMA system is directly attributable to the combined effort of Perception Technology Corporation and our program consultants at Project MAC. At the start of our contract the MACSYMA system was only capable of computing, for a given metric, the inverse metric, Christoffel symbols and Ricci tensors. The latter of these had a bug which was discovered and corrected by Perception Technology Corporation. The list of capabilities as of June 15, 1974 include:

1) For any metric tensor the Ricci tensors and Einstein tensors (covariant or mixed) can be computed in CRE (canonically rational expression) form.

2) For any metric tensor, the inverse metric, Christoffel symbols of both kinds, and mixed Einstein tensors can be computed in CRE or in a truncated power series to arbitrary order in an expansion parameter. We are limited only in the memory of the system.

3) For any metric tensor the components of the full Riemann tensor can be computed in CRE or in a truncated power series.

4) For any metric tensor the components of the Weyl tensor can be computed in CRE or in a truncated power series. We have not attempted classifications yet although this facility is now not difficult.

5) For any metric tensor the equations of motion can be computed in CRE or in a truncated power series.

6) For any metric tensor the d'Alembertian and the t_μ^ν of the new theory, both applied to a scalar field, can be computed.

7) For any metric tensor the components of any second rank tensor in a transformed coordinate system can be computed.

8) For a function $\phi(z-t)$ one has the differential identities $\phi_z \equiv -\phi_t$, $\phi_{zt} \equiv -\phi_{tt}$, $\phi_{zz} \equiv +\phi_{tt}$ which are the wave conditions of certain specially chosen metrics. For components of the metric tensor containing functions of this kind the wave conditions may now be automatically implemented with MACSYMA thus tremendously shortening computations concerned with gravitational radiation.

9) Given a metric tensor in the form $g_{\mu\nu} = \eta_{\mu\nu} + \lambda\phi_{\mu\nu} + \lambda^2\psi_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Lorentz matrix, λ is an expansion parameter, and $\phi_{\mu\nu}$ and $\psi_{\mu\nu}$ are arbitrary tensor fields, MACSYMA is now able to compute the Riemann, Ricci and Einstein tensors and their invariants to second order in λ. This last item was especially important for the completion of the general calculation and represents the only existing tensor manipulation package according to computer experts.

5. APPENDICES

Below are included seven appendices dealing with subjects investigated during the contract period. While these are not strictly of the identical subjects as outlined in the contract, they represent advances in the understanding of the new theory as well as Einsteinian theory. Appendices I, II and V deal with consequences of the general spherically static solution of Einstein's vacuum equations, and each has been submitted for publication. Appendix VI gives the general static solution of the vacuum equations for a metric with plane symmetry. Appendix III deals with Plane Gravitational Waves and represents an extension of a paper read at the American Physical Society Meeting, April 1974. Appendix IV displays some of the interesting properties of exponential matrices. Finally, Appendix VII represents the culmination of our current understanding of the new theory. Efforts will be made to publish as many as possible of these results and appendices in recognized scientific journals and conference proceedings.

APPENDIX I

GENERAL SOLUTION OF EINSTEIN'S EQUATIONS WITH SPHERICAL SYMMETRY*

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ABSTRACT

A general solution for Einstein's theory of gravitation corresponding to spherical symmetry is presented. It is shown that the usual Schwarzchild metrics are members of this general solution. Under the known computational methods believed to be generally covariant some of the member solutions to be exhibited seem to make little or no physical sense. One that seems to be mathematically quite unique and interesting for possible astrophysical applications is described in detail.

By the use of a well-known method of tensor transformation it is proven that the line element

$$ds^2 = fc^2dt^2 - \left(\frac{2m}{1-f}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2) - f^{-1} \left(\frac{2m}{1-f}\right)'^2 dr^2 \qquad (1)$$

where f is a thrice differentiable function of r is a general solution of Einstein's field equations $R_{\mu\nu} = 0$. The usual non-isotropic and isotropic Schwarzchild solutions[1] are members of this general solution with the choices

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$$f = 1 - 2\phi \ , \qquad f = 1 - 2\phi\,(1 + \phi/2)^{-2} \tag{2}$$

where $\phi = m/r$ so that for large r the first order Newtonian correspondence is satisfied. Similarly $f = (1 - \phi)/(1 + \phi)$ yields the well-known Fock solution[2] satisfying the correspondence condition in first order. The solution (1) is new and offers enormous flexibility as we shall see below. (It was originally conjectured during the performance of an ARPA contract directed to study the solutions of author's new theory of gravitation[3], and later explicitly verified by the use of the MACSYMA system of Mathematic's Laboratory, Project MAC, M.I.T.)

Actually it is somewhat disturbing to find that f can be so general as in (1), because we seem to be able to set

$$f = \sum_n a_n \phi^n \tag{3}$$

and produce an arbitrary number of Schwarzchild type singularities, or choose,

$$f = e^{-2\phi} \tag{4}$$

and obtain a solution with no singularity at all, except at $r = 0$. These facts seem to put a question mark on the problem of black holes. Previously we were under the impression that only one legitimate Schwarzchild surface can occur. The solutions corresponding to (3) contradict this contention.

Note that some of the possibilities contained in (3) might also be problematic for either the experimental or the computational aspects of the theory. For example, by setting

$$f = 1 - 2\phi + b\phi^2 \tag{5}$$

$$ds^2 = (1 - 2\phi + b\phi^2)dt^2 - (1 - b\phi/2)^{-2}(r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$$

$$- \frac{1}{f}\,\frac{(1 - b\phi)^2}{(1 - b\phi/2)^4}\,dr^2 \tag{6}$$

where b is arbitrary, one can prove by the use of a standard computational method[4] *(which is believed to be universally applicable) that the three crucial tests of red shift, light bending and perihelion advance are satisfied, but the radar-echo delay depends on b as*

$$\Delta t = 4M\,(\ln 4\,\frac{\ell\ell'}{R^2} + b/2) \tag{7}$$

Since b is arbitrary one may, for instance, set $b = -22$ and get a b-dependent part which cancels the whole gravitational effect expected from the theory. Of course the experiment is more sensitive to the differential delay, $\partial t/\partial R = -8M/R$, but b can always be set arbitrarily large to counter arguments based on the relatively small sensitivity to b. (Units are $c = G = 1$.) This ambiguity is, in fact, a severer form of a well known difficulty already present between the two Schwarzchild solutions $b = 0$ (standard) and $b = 2$ (isotropic) given

by (2). As early as 1959 the present author insisted that the two Schwarzchild solutions might not be physically equivalent, and proposed a test of the isotropy of space using masers.[5] *In 1966 Ross and Schiff exhibited the difficulty more explicitly and offered a possible resolution by bringing in the dynamics of the solar system to bear on the problem.*[6] *But the paradox was really never resolved because the Ross-Schiff argument depended on choosing the standard Schwarzchild metric as the starting point of calculation. If one starts with the Non-isotropic Schwarschild line element and apply the same procedure one can show that the original discrepancy remains*[3]*, namely, the isotropic line element still yields a time-delay 20 μs larger than the standard line element. In (6) this difference is now made to depend on an arbitrary parameter b, such that the discrepancy is* $\delta t = 2M\delta b$ *hence it can be arbitrarily large. A recent investigation*[7] *has further revealed that the choice* $f = 1 - 2\phi + 2\alpha\phi^2 \ln\phi$ *leads, upon the application of the general Robertson-Noonan computational methods*[4]*, to correct first order results for the red shift, light bending and the* $\partial t/\partial R = -8M/R$ *part of the time-delay but gives* α*-dependent differences for the perihelion motion and the absolute value of the gravitational time-delay by the factors* $1 + \alpha/3$ *and* $1 - 3\alpha/2$ *respectively. In view of the claimed covariance of the computational method used this result is very surprising indeed.*

This strange situation may be summed up as follows: "If general relativity is correct all metrics (1) must be observationally equivalent

and any observation made using different frames of reference (1) must be capable of correlating without discrepency." The above discrepencies may then be interpreted in two ways: a) We possess a generally covariant calculational method but general relativity is not correct beyond first order, b) The theory is correct but we do not possess a generally covariant method of calculation. The present writer tends to side with the first interpretation. In two recent articles[8] *it is shown that general relativity conserves rest-mass hence it is generally covariant under a group of transformation (1) that conserves rest-mass. This, however, conflicts with the correspondence limit of special relativity which conserves energy-momentum and not rest-mass. Methods of computation do, however, implicitly imitate special relativity hence presume conservation of energy-momentum. It appears therefore that the clash between the two is reflected in the computed discrepencies above.*

From among the new solutions contained in (1) the possibility of setting or requiring $f = e^{-2\phi}$ *is probably the most interesting for general relativity. In this case we have the line element*

$$ds^2 = e^{-2\phi}dt^2 - \left(\frac{2m}{1-e^{-2\phi}}\right)^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{(2m)^4 e^{-2\phi}}{r^4(1-e^{-2\phi})^4}dr^2 \qquad (8)$$

This new line element has the following remarkable properties: a) For $r >> m$ *it reduces to the usual isotropic Schwarzchild line*

element for all experimental purposes, hence it reproduces all four of the experimental tests [in first and second order (8) is the same as the non-isotropic Schwarzchild line element] b) In the limit of slow motion of the test body the geodesic equations of motion reduce to

$$\frac{d^2x_\mu}{dt^2} = \{{}^{o}_{\mu o}\}\, u_o u^o = -\,\partial_\mu \phi \tag{9}$$

which are exactly the Newtonian equations in spherical coordinates. c) In the limit of slow motion of the source the field equations reduce to

$$\nabla^2\phi = -4\,\pi\sigma \tag{10}$$

which is exactly the Poisson equation. By contrast other line elements reduce to these Newtonian limits in an approximate sense only. d) It is the only solution of Einstein's equations which apply both to the exterior and the interior regions of a mass distribution without a change in form. e) It is harmonic in the sense that if the metric is written in the form, $ds^2 = Dc^2dt^2 - B(r^2d\theta^2 + r^2\sin^2\theta d\phi^2) - Adr^2$, *that is as a factor to special relativity*[8], *then* $\partial_r(\sqrt{-g}\, g^{\mu\nu}) = \partial_r(\sqrt{DB^2A}\, A^{-1}) = 0$. *[in Cartesian coordinates x, y, z (isotropic) the line element is not harmonic]. f) It has no singularity anywhere except at* $r \to 0$ *where it reduces to*

$$ds^2 = -(2m)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{11}$$

The latter represents the surface of an inpenetrable sphere of radius 2m. This spherical surface seems to be suspiciously similar to the Schwarzchild surface but note the difference that (8) has an essential singularity at the surface, not an ordinary one. Furthermore, there is really no interior of this sphere as r starts out at the surface. A possible objection to (8) might be that, like all other solutions of Einstein's theory, it excludes the vacuum gravitational stress-energy tensor t_μ^ν and thereby violates the Newtonian correspondence because we shall be missing the stress-energy tensor

$$t_\mu^\nu = -\partial_\mu \phi \, \partial^\nu \phi + \frac{1}{2} \delta_\mu^\nu \, \partial^\lambda \phi \, \partial_\lambda \phi \tag{12}$$

of the Newtonian theory.

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That the conjectural line element (1) is a general curved-space solution of Einstein's equations was proven by a tensor transformation on the metric. There are also purely flat-space solutions of the form $ds^2 = c^2dt^2 - F^2(d\theta^2 + \sin^2\theta d\phi^2) - F'^2dr^2$ which are not discussed in this paper. These solutions are checked by MACSYMA, a symbolic manipulation system, at Project MAC, MIT. The author is grateful to Dr. Joel Moses and Mr. David Grabel of Project MAC for making this verifi-

cation possible. A direct proof of (1) is found by Dr. Richard Pavelle of Perception Technology Corporation by using the extensive manipulative power of MACSYMA system. Whether these are the only possible general solutions of the theory in spherical coordinates has not yet been established, although a reasoning based on Birkhoff's theorem seems to indicate that there are no others.

REFERENCES

1. *Any text book on General Relativity.*
2. *V. Fock, "The Theory of Space Time and Gravitation", p. 215, Pergamon Press, (1964).*
3. *H. Yilmaz, Phys. Rev. Lettrs., 20, 1399 (1971); Il Nuovo Cimento, 10B, 79 (1972).*
4. *H. P. Robertson and T. W. Noonan, Saunders, pp. 149, 296 (1968). On page 149 the computation method is stated to be applicable for arbitrary interpretation of r and t.*
5. *H. Yilmaz, Phys. Rev. Lettrs., 3, 320 (1959).*
6. *D. K. Ross and L. I. Schiff, Phys. Rev., 141, p. 1216 (1966).*
7. *R. Pavelle and H. Yilmaz, Bull. A.P.S., Spring Meeting, Washington, D. C. (1974).*
8. *H. Yilmaz, Lett. Al Nuovo Cimento, 7, 337 (1973); Annals of Physics, 81, p. 179 (1973).*

APPENDIX II

*DOES GENERAL RELATIVITY PREDICT BLACK HOLES?**

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ABSTRACT

The general static spherically symmetric solution of Einstein's vacuum equations is given. It is shown that besides the non-isotropic Schwarzchild solution which possesses a singularity usually assumed to represent a black hole, we may construct equally valid solutions with a) any number of black holes, b) a black hole of arbitrary radius, c) no black holes. All solutions given in this article except one satisfy the three experimental tests. The arguments presented herein question whether General Relativity does, in fact, unambiguously predict the existence of black holes.

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It is known that the general static and spherically symmetric line element in general relativity is of the form[1]

$$ds^2 = e^{\nu}dt^2 - e^{\mu}r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{\lambda}dr^2 \qquad (1)$$

where ν, μ, λ are functions of r. The vacuum field equations, $R_{\mu\nu} = 0$, of Einstein's theory, however, yields three equations only two of which are independent.

$$R_{11} = \mu'' + \frac{1}{2}\nu'' + \frac{2}{r}\mu' - \frac{1}{r}\lambda' + \frac{1}{2}\mu'^2 - \frac{1}{2}\lambda'\mu' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 = 0 \qquad (2)$$

$$R_{44} = e^{\nu-\lambda}\left(\frac{1}{2}\nu'' + \frac{1}{r}\nu' + \frac{1}{2}\nu'\mu' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2\right) = 0 \qquad (3)$$

It is therefore clear that (1) must contain an arbitrary function, f, of r. Since a general solution containing such an arbitrary function is never written down one usually imposes coordinate conditions such as $\mu = 0$ (standard), $\mu - \lambda = 0$ (isotropic), or $\partial_\nu(\sqrt{-g}\, g^{\mu\nu}) = 0$ (harmonic) and obtains special line elements. All the solutions so far obtained by such specializations have a common property in that, by a method of calculation believed to be applicable to any line element with arbitrary interpretations of t and r, they yield the three crucial tests of red shift, light bending and perihelion advance within experimental accuracy[2].

We have recently determined[3] the general solution of the differential equations (2) and (3). The form of the line element (1) containing an arbitrary function, f, is found to be

$$ds^2 = fdt^2 - \left(\frac{2m}{1-f} \right)^2 (d\theta^2 + \sin^2\theta d\phi^2) - \frac{1}{f} \left[\left(\frac{2m}{1-f} \right)' \right]^2 dr^2 \qquad (4)$$

This general solution, which can be checked by using equations (2) and (3) yields the above specializations directly by specifying f as $f = 1 - 2\phi$, $f = 1 - 2\phi(1 + \phi/2)^{-2}$, $f = 1 - \phi(1 + \phi)^{-1}$ where $\phi = m/r$. What is found, by a general application of (4) is, however, quite disturbing. We shall first give a few examples to indicate the nature of our worries and later comment on the possible implications of the situation for the theory as a whole.

i) All the special solutions so far known involve only one singularity of the Schwarzchild type. But consider the choices

$$e^{\nu} = f = \prod_{j=1}^{p} (1 - C_j\phi) \quad , \qquad \sum_{j}^{p} C_j = 2 \qquad (5)$$

where $C_j > 0$. These solutions are exact, satisfy the boundary conditions at infinity, reduce to the Newtonian theory in first order and yield the three crucial tests correctly but in general possess p (an arbitrary number) singularities. If one computes the location of the null hypersurfaces corresponding to these singularities[3] it is found that these singularities

satisfy the usual definition of black holes. However, we can have p such singularities, p being any number. Which one of these singular surfaces will be identified with the physical black hole assumed to exist in the literature?

ii) Another choice which satisfies the above desiderata

$$e^{\nu} = f = (1 - 2\xi\phi)^{1/\xi} \tag{6}$$

where ξ is a positive number. In this case we have only one singularity which corresponds to a null hypersurface but its position

$$r = 2m\xi \tag{7}$$

can be anywhere. For $\xi = 2.5 \times 10^5$ the singularity will engulf the sun without affecting the three crucial tests theoretically. Is the sun a black hole?

iii) Consider now ξ to be negative, namely,

$$e^{\nu} = f = (1 + 2\xi\phi)^{-1/\xi} \tag{8}$$

in which case there exists no singularity at all except for $\xi = 0$, $r = 0$. Note that (8) is not a non-legitimate solution. A special case $\xi = 1$ of (8) could have been obtained from (2) and (3) by setting $\nu + \lambda = 0$. This condition indeed yields $\xi = \pm 1$ hence $f = (1 \mp 2\phi)^{\pm 1}$ one member of which is exactly the "standard" line element.

The other member is just as un-objectionable and yields the three experimental tests correctly. However, it has no singularity and no black hole!

iv) Consider now the more interesting possibility[3]

$$e^{\nu} = f = 1 - 2\phi + 2\alpha\phi^2 \ln \phi \tag{9}$$

where α *is an arbitrary number. This line element satisfies the Newtonian limit and the boundary conditions at infinity and, by the same general methods of calculation mentioned above, yields correct values for the first order tests of red shift and light deflection. For the second order test, namely, the perihelion advance, we have*

$$\Delta\phi = \frac{6\pi m^2}{h^2} \left(1 + \frac{\alpha}{3}\right) \tag{10}$$

which is α*-dependent. Does this mean that the method of calculation is not as general as it was supposed to be, or does it mean that general relativity is an adjustable-parameter theory in disguise? Furthermore, assuming that (9) were the only line element we knew (other being assumed not yet discovered), how does one calculate a black hole behavior free from ambiguities? In view of all the above does the usual "null surface" characterization*[4] *of a black hole have an invariant meaning?*

v) Let us construct the Kretchmann invariant for (4),

$$I = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = (12/m^4)(1 - f)^6 \qquad (11)$$

This appears to assign a special significance to the "standard" line element as $I = 12m^2/r^6$ is a simple power of r only in that case. But if this is of any invariant significance why does (11) not possess a singularity at $r = 2m$ as an invariant characterization of a black hole? Note also that it is not possible to say that the new singularities we presented in item i) are merely coordinate singularities, hence they should be forgotten. Because in view of all that is shown we can equally well argue that the usual singularity $r = 2m$ is itself a coordinate singularity hence it is to be forgotten. The conclusion seems to be that despite the voluminous literature build up around the mysterious concept of a black hole, Einstein's theory does not really predict a unique behavior for such an object.

These and other examples that one can easily construct [for instance $e^{\nu} = f = 1 - 2\phi \prod_{j=1}^{p} (1-B_j\phi)$, $1 - f = 2\phi \prod_{j=1}^{p} (1 - B_j\phi)$, $B_j > 0$] on the basis of the general solution (4) seem to lead to a necessity of examining anew the status of the problem of coordinates in general relativity.[5] One could say that due to the coordinate invariance any calculation carried out in the standard line element can be taken over into others (by transformation) in the sense of redefining the integration variables and their limits. But the problem is that we do not know the meaning of r in any of these coordinates (including the

standard coordinates) and we do not know the integrations limits. When we interpret r as the radius in the standard line element we assign a number to the distance. If we interpret r as the distance in another solution we should assign a different number for the distance so as to find the end results of the calculations the same. That much is clear and trivial. If we say that the correct assignment of a number is possible only in the standard solution then either we are attributing an ad hoc privilege to the standard solution or we must have a reason for the correctness of this assignment which should then be generalizable to other solutions. In special relativity, from which the general theory is an extension, such a problem would be resolved by the constancy of the velocity of light, $|dr| = c|dt|$, because we then have an independent way of measuring r by light signals in "every" Lorentz line-element. The Lorentz line element, however, is not generally covariant (for example we do not use such otherwise covariant substitutions $x' = x + Vt$, $t' = t$) but covariant only under such substitutions which leave the velocity of light invariant. It seems to follow that a similar imposition on (4) must be made to identify physically the meaning of r by an independent physical means, at least in one system of coordinates. The general theory of relativity does not seem to satisfy this requirement although everyone believed that it does.[6]

The conclusion seems to be that, in the past, special non-covariant, choices such as $\mu = 0$, $\mu - \lambda = 0$, $\partial_\nu(\sqrt{-g}\, g^{\mu\nu}) = 0$ tended to yield at

least a plausible way of imposing the extra requirement since they eliminated the more troublesome choices of the type (9). Similarly the usual PPN method[7] *which relies only on power series expandable metrics imposes an extra constraint and disregards (9) and other non-power series expandable solutions. Under any of the above conditions [including $\nu + \lambda = 0$ where $|\xi| = 1$ or even the more general case $|\xi| \lesssim 2.5 \times 10^5$], the metric (4) yields the three experimental tests to the accuracy of the present observations but the other possible choices are still too large in number, physically non-equivalent to each other, and, do not answer unambiguously the original question we posed: Does general relativity unambiguously predict the existence of black holes?*

REFERENCES

1. *Any book in general relativity, e.g., Reference (2).*
2. *H. P. Robertson and T. W. Noonan, Saunders, pp. 149, 296 (1968).*
3. *R. Pavelle and H. Yilmaz, Washington Meeting, APS, B, (1974).*
4. *D. Finkelstein, Phys. Rev., 110, 965 (1958).*
5. *E. P. Wigner, Rev. Mod. Phys., 29, 255 (1957).*
6. *H. Yilmaz, Ann. Phys., 81, 179 (1974) gives a new theory which satisfies such a condition and predicts a unique behavior for a collapsed object. This behavior is characterizable as a "gray hole" not a black hole. See also N. Rosen, Lettere Al Nuovo Cimento, 8, 889 (1973).*
7. *C. M. Will, Physics Today, September (1972).*

APPENDIX III

PLANE GRAVITATIONAL WAVES

a) Plane Waves in Einstein's Theory

It is well known that the linearized (first order) Einstein theory has plane gravitational waves of the form[1]

$$g_{00} = 1 \quad , \quad g_{33} = -1 \tag{1}$$

$$g_{11} = -1 + 4Q \qquad g_{22} = -1 - 4Q, \; g_{12} = R \tag{2}$$

where Q and R are arbitrary functions of t - z

$$Q = Q(t-z) \; , \; R = R(t - z) \tag{3}$$

satisfying $\Box^2 Q = 0$, $\Box^2 R = 0$. It must be stressed at once that for the above metric $\Box^2 = D^\alpha D_\alpha$ reduces to $\Box^2 = (\partial^2/\partial t^2 - \partial^2/\partial x^2)$ hence Q and R are solutions of the d'Alembert equation of the curved space in question. Note further that forms such as $Q = t - z$, or $Q = (t - z)^\alpha$ are not acceptable as waves since they do not satisfy appropriate wave conditions. The solution (2) is, as is well known, the basis of the usual calculations of the energy radiation[2] and the elaborate experimental investigations to detect such radiation.[3]

In the course of our investigations it became clear, however, that the solution (2) and, together with it, the usual belief that Einstein's theory would predict gravitational radiation of energy gets into a peculiar difficulty in second order. To see the nature of the difficulty we let

$$g_{11} = -1 + 4\lambda Q + \lambda^2 U \tag{4}$$

$$g_{22} = -1 + 4\lambda Q + \lambda^2 V \tag{5}$$

$$g_{12} = 4\lambda R + \lambda^2 W \tag{6}$$

where λ is an expansion parameter and U, V, W are quantities that are of second order. Substituting into the Einsteinian equations

$$R_{\mu\nu} = 0 \tag{7}$$

one finds that the second order quantities U and V must satisfy the equation

$$(U + V)'' + 32(QQ'' + RR'') + 16(Q'^2 + R'^2) = 0 \tag{8}$$

where prime (') implies derivative with respect to t or z or $\xi = t - z$. (note that $\Box^2 f = 0$ is not equivalent to $f'' = 0$.) Interestingly W does not appear in the equation (8) showing that the non-linear term in g_{12} is of third order or higher. Being a differential equation in Q and R equation (8) implies that U + V is expressible as a function of Q and R. The most general second order expression is

$$U + V = \alpha Q^2 + \beta QR + \gamma R^2 \tag{9}$$

Substituting into (8) one sees first of all that $\beta = 0$. Furthermore

$$(\alpha+16)QQ'' + (\alpha+8)Q'^2 + (\gamma+16)RR'' + (\gamma+8)R'^2 = 0 \tag{10}$$

Since Q and R are linearly independent functions it is sufficient to analyze the case of Q alone (or R alone). We thus set

$$(\alpha + 16)QQ'' + (\alpha + 8)Q'^2 = 0 \tag{11}$$

A trivial solution is $Q'' = 0$, $Q = t - z$ with $\alpha = -8$. But this is clearly unacceptable because $Q = t - z$ is not a wave. (If $Q = t - z$ were acceptable then even the linearized solution (2) is not unique, since $g_{11} = -1 + aQ$, $g_{22} = -1 + bQ$, where a and b arbitrary, would be acceptable). The general solution of (11) is

$$Q = C \cdot (t - z)^{\frac{16+\alpha}{24 + 2\alpha}} \tag{12}$$

This Q although satisfies the d'Alembert equation $\Box^2 Q = 0$ is nevertheless unacceptable as a gravitational wave because it is prescribed (Q cannot carry information) and will not satisfy wave conditions at infinity. The only other way to obtain a solution is to set in (10) $R = iQ$ but this makes R and Q linearly dependent and the metric non-real. The conclusion seems to be that in Einstein's theory the linearized solution (2) does not extend to second order consistently unless $Q = R = 0$, that is the flat metric.

b) Plane Waves in the New Theory

In the new theory this same problem goes as follows: The metric is[4]

$$g_{\mu\nu} = \eta_{\mu\nu}\, e^{2(\phi - 2\tilde{\phi})} \tag{13}$$

$$\tilde{\phi} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & Q & R \\ \cdot & \cdot & R & -Q \end{bmatrix} \tag{14}$$

where $\phi = \phi^{\mu}_{\mu} = 0, \Box^2\tilde{\phi} = 0$. Expansion of (13) yields

$$g_{11} = -\cosh 4\Omega + \frac{Q}{\Omega}\sinh 4\Omega \tag{15}$$

$$g_{22} = -\cosh 4\Omega - \frac{Q}{\Omega}\sinh 4\Omega \tag{16}$$

$$g_{12} = \frac{R}{\Omega}\sinh 4\Omega \tag{17}$$

$$g_{00} = -g_{33} = 1 \tag{18}$$

where $\Omega = (R^2 + Q^2)^{1/2}$. This metric is proven to be an exact solution of the field equations of the new theory. The wave equation is indeed satisfied, as stated, namely, without becoming prescribed and the stress-energy tensor is the canonical stress-energy tensor of the Lagrangian

$$L_f = -\partial^\lambda \phi^\alpha_\beta \, \partial_\lambda \phi^\beta_\alpha + \frac{1}{2} \partial^\lambda \phi \, \partial_\lambda \phi \tag{19}$$

Since $\phi = 0$ this yields

$$t^\nu_\mu = -2 \, \partial_\mu \phi^\alpha_\beta \, \partial^\nu \phi^\beta_\alpha + \delta^\nu_\mu \partial^\lambda \phi^\beta_\alpha \, \partial_\lambda \phi^\alpha_\beta \tag{20}$$

For R = 0 the solution reduces to the form

$$g_{11} = -e^{-4Q} \quad , \quad g_{22} = -e^{4Q} \tag{21}$$

whereas for Q = 0 to the form

$$g_{11} = g_{22} = - \cosh 4R, \quad g_{12} = \pm \sinh 4R \tag{22}$$

We conclude that in the new theory the linearized solution (2) extends consistently to second order. The t^ν_μ then provides the physical basis of energy radiation and it can be shown that the amount radiated is[5]

$$- \frac{\partial E}{\partial t} = \frac{16}{5C^5} \dddot{D}_{\mu\nu} \dddot{D}^{\mu\nu} \tag{23}$$

where $D_{\mu\nu}$ is the quadrupole moment of the matter distribution. The implication of this Appendix seems to be that a corresponding calculation in Einstein's theory cannot be carried out consistently because the first order plane wave solutions on which the computation depends do not carry over to second order. The second order consistency is on the other hand absolutely necessary because equations themselves

are non-linear and the stress-energy tensor, t_μ^ν, is a second order quantity. Consequently if experimentalists discover the existence of the gravitational waves this could be regarded as a confirmation of the new theory.

Interestingly Einstein himself seems to have been aware of the non existence of gravitational radiation in his theory. In an undated letter (1936-37) to Max Born he states:[6] "Together with a young collaborator, I arrived at the interesting result that the gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows that the nonlinear general relativistic field equations can tell us more or, rather, limit us more than we have believed up to now." Unfortunately later investigators do not seem to have paid attention to Einstein's conclusion. They have indeed produced exact solutions to the equations with t - z dependence but they did not realize that these solutions are prescribed. A familiar example they cite is of the form[7]

$$g_{11} = -e^{4K - 4Q} \qquad g_{22} = -e^{4K + 4Q} \tag{24}$$

where K is a second order quantity ($L = e^{4K}$ is unity in first order). Expanding as in (1) (2) and (3) one finds (K^2 is fourth order hence neglected)

$$U = V = -8Q^2 - 4K \tag{25}$$

Letting $K = \epsilon Q^2$ one finds $\alpha = -16 - 8\epsilon$, hence the problem is reduced to the previous case of (12). In particular if $\epsilon = -1$, $\alpha = -8$, hence $Q = C \cdot (t - z)$. This case is not only prescribed but also turns out to be flat, namely, $R^{\sigma}_{\alpha\beta\gamma} = 0$.

Another example cited is the metric[8]

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\xi \, \partial_\nu\xi \, [(x^2 - y^2)G(\xi) + xy \, H(\xi)] \quad (26)$$

where $G(\xi)$ and $H(\xi)$ are arbitrary functions of $\xi = t - z$. The problem with this solution is that when explicitly written out it yields

$$g_{00} = 1 + (x^2 - y^2) \, G + xy \, H = 1 + \Lambda \quad (27)$$

$$g_{03} = -\Lambda \quad , \quad g_{33} = -1 + \Lambda \quad (28)$$

namely, it depends only on a single function Λ, hence g_{00}, g_{03}, g_{33} are not linearly independent. To represent a spin 2 graviton propagating with the velocity of light one however needs at least two linearly independent field variables. It will not do to say that G and H are linearly independent because they do not contribute to $g_{\mu\nu}$ in a linearly independent way. At the present time we are not aware of the existence of a solution of Einstein's theory which can be considered physically as a gravitational wave.

REFERENCES

1. H. P. Robertson and T. W. Noonan, "Relativity and Gravitation", W. B. Saunders Company, p. 261 (1969).

2. Landau, L. D., and Lifshitz, E. M., "The Classical Theory of Fields", Addison-Wesley Publishing Company, p. 366, (1962).

3. Weber, J. "General Relativity and Gravitation Waves". Interscience Publishers Inc. (1961).

4. Yilmaz, H., Ann. Phys., 81, 179 (1973).

5. Yilmaz, H., Nuovo Cimento, 10B, 79 (1972).

6. Born, M. and Born, H., "The Born Einstein Letters", Walker and Company, p. 125 (1971).

7. Misner, C., Thorne, K. S., and Wheeler, J. A., "Gravitation", W. H. Freeman and Company, p. 958 (1973).

8. Sachs, R. K., "Relativity Theory and Astrophysics", J. Ehler, Ed. Am. Math. Soc., Providence, R.I., p. 130 (1967).

APPENDIX IV

EXPONENTIAL MATRICES

In the New Theory, for particular physical problems, functional expansions of the form $e^{\tilde{\lambda}} = \tilde{1} + \tilde{\lambda} + \frac{\tilde{\lambda}^2}{2!} + \ldots$ where $\tilde{\lambda}$ is a square matrix are considered. Here $\tilde{\lambda}^2$ is the product matrix. The MACSYMA system is ideal for generating useful closed form expansions of this kind and we have been able to find many interesting cases. The convergence of the following forms may be proved by induction. In the following examples the dot represents a zero element.

$$\lambda = \begin{pmatrix} \cdot & A & B \\ -A & \cdot & C \\ -B & -C & \cdot \end{pmatrix} \tag{1}$$

$$e^{\lambda} = \begin{pmatrix} 1 + \frac{(A^2+B^2)(\cos\rho-1)}{\rho^2} & \frac{A\sin\rho}{\rho} + \frac{BC(\cos\rho-1)}{\rho^2} & \frac{B\sin\rho}{\rho} - \frac{AC(\cos\rho-1)}{\rho^2} \\ -\frac{A\sin\rho}{\rho} + \frac{BC(\cos\rho-1)}{\rho^2} & 1 + \frac{(A^2+C^2)(\cos\rho-1)}{\rho^2} & \frac{C\sin\rho}{\rho} + \frac{AB(\cos\rho-1)}{\rho^2} \\ -\frac{B\sin\rho}{\rho} - \frac{AC(\cos\rho-1)}{\rho^2} & -\frac{C\sin\rho}{\rho} + \frac{AB(\cos\rho-1)}{\rho^2} & 1 + \frac{(B^2+C^2)(\cos\rho-1)}{\rho^2} \end{pmatrix} \tag{2}$$

where $\rho^2 = A^2 + B^2 + C^2$.

$$\lambda = \begin{pmatrix} \cdot & \cdot & \cdot & R \\ \cdot & \cdot & \cdot & S \\ \cdot & \cdot & \cdot & T \\ -R & -S & -T & \cdot \end{pmatrix} \tag{3}$$

$$e^{\lambda} = \begin{pmatrix} 1 + \frac{R^2}{\rho^2}(\cos\rho - 1) & \frac{RS}{\rho^2}(\cos\rho - 1) & \frac{RT}{\rho^2}(\cos\rho - 1) & \frac{R}{\rho}\sin\rho \\ \frac{RS}{\rho^2}(\cos\rho - 1) & 1 + \frac{S^2}{\rho^2}(\cos\rho - 1) & \frac{ST}{\rho^2}(\cos\rho - 1) & \frac{S}{\rho}\sin\rho \\ \frac{RT}{\rho^2}(\cos\rho - 1) & \frac{ST}{\rho^2}(\cos\rho - 1) & 1 + \frac{T^2}{\rho^2}(\cos\rho - 1) & \frac{T}{\rho}\sin\rho \\ -\frac{R}{\rho}\sin\rho & -\frac{S}{\rho}\sin\rho & -\frac{T}{\rho}\sin\rho & \cos\rho \end{pmatrix} \tag{4}$$

where $\rho^2 = R^2 + S^2 + T^2$.

$$\lambda = \begin{pmatrix} Q & R \\ R & Q \end{pmatrix} \qquad e^{\lambda} = \begin{pmatrix} \cosh(R) & \sinh(R) \\ \sinh(R) & \cosh(R) \end{pmatrix} \cdot e^{Q} \tag{5}$$

$$\lambda = \begin{pmatrix} A & \cdot & \cdot & \cdot \\ \cdot & B & \cdot & \cdot \\ \cdot & \cdot & C & \cdot \\ \cdot & \cdot & \cdot & D \end{pmatrix} \qquad e^{\lambda} = \begin{pmatrix} e^{A} & \cdot & \cdot & \cdot \\ \cdot & e^{B} & \cdot & \cdot \\ \cdot & \cdot & e^{C} & \cdot \\ \cdot & \cdot & \cdot & e^{D} \end{pmatrix} \tag{6}$$

$$\lambda = \begin{pmatrix} Q & R \\ R & -Q \end{pmatrix} \tag{7}$$

$$e^{\lambda} = \begin{pmatrix} \cosh(R^2+Q^2)^{1/2} + \dfrac{Q\sinh(R^2+Q^2)^{1/2}}{(R^2+Q^2)^{1/2}} & \dfrac{R\sinh(R^2+Q^2)^{1/2}}{(R^2+Q^2)^{1/2}} \\ \dfrac{R\sinh(R^2+Q^2)^{1/2}}{(R^2+Q^2)^{1/2}} & \cosh(R^2+Q^2)^{1/2} - \dfrac{Q\sinh(R^2+Q^2)^{1/2}}{(R^2+Q^2)^{1/2}} \end{pmatrix} \tag{8}$$

APPENDIX V

GENERAL STATIC CURVED SOLUTIONS OF EINSTEIN'S EQUATIONS FOR SPHERICALLY SYMMETRIC METRICS*

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For a general, static, spherically symmetric coordinate system we find the general curved solutions of the Einstein equations $G^{\alpha}_{\beta} = 0$ and $G^{\alpha}_{\beta} = \lambda\delta^{\alpha}_{\beta}$. From these general solutions the well known standard solutions are easily generated.

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INTRODUCTION

In a static spherically symmetric coordinate system there are a number of well known particular solutions of Einstein's equations corresponding to specific choices of the energy-momentum tensor. However, the problem of finding the general solution of the Einstein equations corresponding to a particular matter distribution does not seem to have been considered. It could be argued that in view of Birkhoff's theorem[1] this problem would be of marginal interest. This is not the case, however. On the one hand the various proofs of the Birkhoff theorem do not specify a method for constructing the most general form of the metric. Secondly, it is of mathematical interest to find general solutions for any system of non-linear differential equations as an existence proof might miss some particular solutions. Finally, although the general solution must itself be a transform of some particular solution[2], there is no a priori guarantee that all solutions, for a given matter distribution will, give the identical predictions of physical phenomena. This final point is discussed elsewhere[3] where we show that certain transforms of the Schwarzchild solution do not seem to lead, unambigously, to the same physical predictions.

Below, we solve the Einstein equations for a spherically symmetric coordinate system. We consider both the vacuum equations and the equations with the cosmical constant. From these general solutions certain well known solutions are easily generated.

All mathematical statements in this paper have been verified and intermediate steps simplified using the symbolic manipulation system, MACSYMA, developed by the Math Lab group at Project MAC, M.I.T.

1. VACUUM EQUATIONS

Consider the spherical coordinate system in which the line element may be written in the form[4]

$$ds^2 = - A\,dr^2 - F(d\theta^2 + \sin^2\theta d\phi^2) + D\,dt^2 \quad , \qquad (1.1)$$

where A, F, D are radial functions of class C^2. We wish to find the most general relation between A, F and D so that (1.1) satisfies the Einstein vacuum field equations. We shall not be interested in cases in which A, F or D themselves vanish everywhere.

It can be shown that for the metric (1.1) the only non vanishing components of the Einstein tensor are

$$G^1_1 = \frac{D[4AF - (F')^2] - 2FF'D'}{4AF^2D} \qquad (1.2)$$

$$G^2_2 = G^3_3 = -\frac{D^2[A(2FF''-(F')^2)-A'FF']+D[A(D'FF'+2D''F^2)-A'D'F\ ]-AF(D')^2}{4A^2F^2D^2} \qquad (1.3)$$

$$G^4_4 = \frac{4A^2F \;\; + 2A'FF' + A[(F')^2 - 4FF'']}{4A^2F^2} \quad . \qquad (1.4)$$

These components set to zero yield the differential equations we wish to solve.

We approach this system of equations by noticing that $G_1^1 = 0$ may be solved for A as

$$A = \frac{F'(DF' + 2D'F)}{4DF} \quad . \tag{1.5}$$

We substitute this into (1.3) and (1.4) (set to zero) to find

$$G_2^2 = G_3^3 = \frac{D'F(2D'FF'' - 3D'(F')^2 - 2D''FF')}{(F')^2(DF' + 2D'F)^2} = 0 \tag{1.6}$$

and

$$G_4^4 = -\frac{2D(2D'FF'' - 3D'(F')^2 - 2D''FF')}{(F')(DF' + 2D'F)^2} = 0. \tag{1.7}$$

It is clear that if $D \neq 0$ and $F \neq 0$ the most general way we may satisfy (1.6) and (1.7) simultaneously is by requiring

$$2D'FF'' - 3D'(F')^2 - 2D''FF' = 0 \quad . \tag{1.8}$$

This differential equation is easily integrated to yield

$$D = K_1 - K_2F^{-1/2} \tag{1.9}$$

where K_1 and K_2 are constants.

In (1.6) and (1.7) we note that $F' \neq 0$ and $DF' + 2D'F \neq 0$. These "restrictions" on the functions are, from (1.5), nothing more than the requirement $A \neq 0$ which we stipulated above.

We now substitute (1.9) into (1.5) to obtain

$$A = \frac{K_1 (F')^2}{4F \cdot (K_1 - K_2 F^{-1/2})} \tag{1.10}$$

From (1.9) and (1.10) in addition to a trivial coordinate transformation we then have[5]

<u>THEOREM</u>: <u>The metric tensor defined by the line element</u>

$$ds^2 = -\frac{(F')^2}{4F(1 - KF^{-1/2})} dr^2 - F(d\theta^2 + \sin^2\theta d\phi^2) + (1 - KF^{-1/2}) dt^2 , \tag{1.11}$$

<u>satisfies the Einstein vacuum equations identically where F is an arbitrary function of r of class C^2.</u>

We may transform (1.11) into a more intuitive form by letting $F \to B^2 r^2$ where $B = B(r)$ is arbitrary. With this new definition it is easily seen that

$$ds^2 = -\frac{[(Br)']^2}{1 - \frac{K}{Br}} dr^2 - B^2(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + (1 - \frac{K}{Br}) dt^2 . \tag{1.12}$$

The non-isotropic Schwarzchild solution[6] may now be obtained by setting B = 1 to find

$$ds^2 = -\frac{1}{1-\frac{K}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1-\frac{K}{r})dt^2 \ . \tag{1.13}$$

From (1.11) we may also obtain Fock's solution[7]. By choosing $F = (r + K/2)^2$ we find

$$ds^2 = -\frac{r+\frac{K}{2}}{r-\frac{K}{2}}dr^2 - (r+\frac{K}{2})^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{r-\frac{K}{2}}{r+\frac{K}{2}}dt^2. \tag{1.14}$$

From (1.11) or (1.12) we may find the isotropic spherically symmetric line element by setting

$$\frac{(F')^2}{4F(1-KF^{-1/2})} = \frac{F}{r^2} \tag{1.15}$$

This differential equation may easily be solved for F and we find

$$F = \frac{(K_1 + \frac{K}{2r})^4 r^2}{4K_1^2} \tag{1.16}$$

where K_1 is an integration constant. Thus, the isotropic line element takes the form

$$ds^2 = -\left(1 + \frac{K}{4r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \frac{\left(1 - \frac{K}{4r}\right)^2}{\left(1 + \frac{K}{4r}\right)^2} dt^2 \qquad (1.17)$$

in accordance with the well known result.[8]

2. EQUATIONS WITH THE COSMICAL TERM

For the metric (1.1) it follows from (1.2), (1.3) and (1.4) that the component form of the Einstein equation

$$G^i_j = \lambda\delta^i_j \quad , \qquad (2.1)$$

where λ is the "cosmical constant" is

$$G^1_1 = \frac{D(4AF - (F')^2) - 2FF'D'}{4AF^2D} = \lambda \qquad (2.2)$$

$$G^2_2 = G^3_3 = \frac{D^2[A(2FF''-(F')^2)-A'FF']+D[A(D'FF'+2D''F^2)-A'D'F^2]-AF^2(D')^2}{4A^2F^2D^2} = \lambda \quad (2.3)$$

$$G^4_4 = -\frac{4A^2F^2 + 2A'FF' + A((F')^2 - 4FF'')}{4A^2F^2} = \lambda \quad . \qquad (2.4)$$

As we did for the more simple case we solve (2.2) for A to find

$$A = -\frac{F'}{4FD}\left(\frac{DF' + 2D'F}{F\lambda - 1}\right) \qquad (2.5)$$

We substitute this into (2.3) and (2.4). In both cases we then subtract (the right hand side) from the left to obtain differential equations which turn out to be similar to each other. This procedure gives

$$G_2^2\Big|_{2.5} - \lambda = \frac{2D\{(2D'F^2F''+D(F')^3-D'F(F')^2-2D''F^2F')\lambda-2D'FF''+3D'(F')^2+2D''FF'\}}{F'(DF' + 2D'F)^2} = 0 \tag{2.6}$$

and

$$G_4^4\Big|_{2.5} - \lambda = -\frac{D'F\{(2D'F^2F''+D(F')^3-D'F(F')^2-2D''F^2F')\lambda-2D'FF''+3D'(F')^2+2D''FF'\}}{(F')^2(DF' + 2D'F)^2} = 0 \tag{2.7}$$

It is clear that to solve (2.6) and (2.7) we need to find the relation between F and D which causes the expression in curly brackets to vanish. For $\lambda = 0$ we know that the relation we seek must reduce to

$$D = K_1 - K_2F^{-1/2} \tag{2.8}$$

hence we now seek a function N(F) such that

$$D = K_1 - K_2F^{-1/2} + \lambda N(F) \tag{2.9}$$

satisfies (2.6) and (2.7).

Substitution of (2.9) into (2.6) results, after some laborious calculation, is the following differential equations for N:

$$N - N'F = 0$$
$$3N' - K_1 = 0 \tag{2.10}$$

These are easily integrated to find

$$N = -\frac{K_1 F}{3} \tag{2.11}$$

Thus, (2.9) becomes

$$D = K_1(1 - \frac{\lambda F}{3}) - K_2 F^{-1/2} \tag{2.12}$$

With this expression for D we find

$$A = \frac{K_1 (F')^2}{4(K_1 - K_2 F^{-1/2} - \frac{\lambda}{3} K_1 F)F} \tag{2.13}$$

From (2.1:) and (2.14) and a trivial coordinate transformation to absorb an integration constant we have[5]

THEOREM: The metric defined by

$$ds^2 = -\frac{(F')^2}{4 \cdot F(1 - KF^{-1/2} - \frac{\lambda}{3} F)} dr^2 - F(d\theta^2 + \sin^2\theta d\phi^2) + (1 - KF^{-1/2} - \frac{\lambda}{3}F)dt^2$$

(2.14)

satisfies

$$G^{\mu}_{\nu} = \lambda\delta^{\mu}_{\nu} \tag{2.15}$$

for any function F(r) of class C^2.

Let us now choose $F = r^2$ in (2.14). With this choice we find

$$ds^2 = -\frac{dr^2}{1 - \frac{K}{r} - \frac{\lambda}{3} r^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1 - \frac{K}{r} - \frac{\lambda}{3} r^2)dt^2. \tag{2.16}$$

This is the Kottler solution[9] and represents one of the very few known (if indeed others have been found) solutions of (2.1).

We now derive an expression for the isotropic solution corresponding to (2.14) by setting

$$\frac{(F')^2}{4F(1 - KF^{-1/2} - \frac{\lambda}{3} F)} = \frac{F}{r^2} \tag{2.17}$$

We see that F must satisfy

$$\int \frac{dF}{F(1 - KF^{-1/2} - \frac{\lambda}{3} F)^{1/2}} = \ln c \cdot r^{\pm 2} \tag{2.18}$$

where C is an integration constant. This integral does not appear to be expressible in closed form. However, one can obtain expression for particular cases of interest. By setting $K = 0$ in (2.18) we may perform the integration and solve for F to find

$$F = \frac{12\ CR^2}{(\lambda R^2 + C)^2} \tag{2.19}$$

Then

$$ds^2 = -\frac{12\ C}{(\lambda R^2 + C)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \left(1 - \frac{\lambda CR^2}{(\lambda R^2 + C)^2}\right)dt^2 \tag{2.20}$$

Here we have the isotropic static form of the solution of (2.1). Other solutions of possible cosmological interest could be found by performing a series expansion on the left hand side of (2.18). We have not considered such cases at the present time.

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REFERENCES

1. See, for example, W. B. Bonner, "Recent Developments in General Relativity", Pergamon Press, (1962). Page 167 - The theorem in Bonner's form states that all spherically symmetric solutions of $G^i_j = \lambda\delta^i_j$ may be transformed into the appropriate Schwarzchild solution (2.16).

2. There is no known method for generating the general solution from a particular solution via a coordinate transformation.

3. H. Yilmaz and R. Pavelle, to appear.

4. In this form the differential equations take a more simple form.

5. This theorem may also be proved by noting that since (1.11) may be transformed into the Schwarzchild solution (1.13), Birkhoff's theorem is satisfied (Reference 1) and it follows that (1.11) is a general solution in these coordinates. A similar argument applies to (2.14

6. R. C. Tolman, "Relativity, Thermodynamics and Cosmology", Oxford at the Clarendon Press, 1962, p. 82.

7. V. Fock, "The Theory of Space Time and Gravitation", Pergamon Press, 1964, p. 58.

8. Reference 6, (82.14).

9. F. Kottler, "Uber die physikalischen Grundlagen der Einsteinschen Gravitations theory., Ann. Physic (Ser. 4), p. 56, (1918), 401.

APPENDIX VI

SOLUTIONS WITH PLANE SYMMETRY IN EINSTEIN'S THEORY

Consider the Cartesian-like coordinate system defined by

$$ds^2 = - F(dx^2 + dy^2) - Hdz^2 + K\,dt^2 \tag{1.1}$$

where F, H and K are function of z. We wish to find the general relations between these functionals so that (1.1) satisfies Einstein's Vacuum Equation:

For (1.1) we find the components of the Einstein tensor (set to zero) to be

$$G^1_1 = G^2_2 = - \{K(H[2F^2K'' + FF'K'] - F^2H'K') - F^2H(K')^2$$
$$+ K^2(H(2FF'' - (F')^2) - FF'H')\} / 4F^2H^2K^2 = 0 \tag{1.2}$$

$$G^3_3 = - \frac{F'(2FK' + KF')}{4F^2HK} = 0 \tag{1.3}$$

$$G^4_4 = - \frac{H(F[2F^2F''+F(F')^2] + F^2(2FF''-(F')^2) - F^2(F')^2)-2F^3F'H'}{4F^4H^2} \tag{1.4}$$

We may satisfy (1.3) by requiring

$$2FK' + KF' = 0 \tag{1.5}$$

This differential equation may be integrated and we find

$$F = C_1/K^2 \qquad (1.6)$$

where C_1 is a constant. This relation implies that (1.2) and (1.4) become

$$G^1_1 = G^2_2 = \frac{H(2KK'' - 5(K')^2) - KH'K'}{4H^2K^2} = 0 \qquad (1.7)$$

$$G^4_4 = \frac{H(2KK'' - 5(K')^2) - KH'K'}{H^2K^2} = 0 \qquad (1.8)$$

Hence, by solving (1.7) or (1.8) we will have a general solution of the Einstein Vacuum Equation. These are integrable and we find that

$$H = C_2 \frac{(K')^2}{K^5} \qquad (1.9)$$

satisfies (1.8) and (1.8). Thus, the metric tensor components defined by

$$ds^2 = -\frac{C_1}{K^2}(dx^2 + dy^2) - C_2 \frac{(K')^2}{K^5} dz^2 + K\, dt^2 \qquad (1.10)$$

is the general static solution of $G^i_j = 0$ for any function $K(z) \neq 0$.

On the other hand, we may satisfy (1.3) by choosing $F' = 0$. With this choice it is found that

$$G^1_1 = G^2_2 = - \frac{2KH\,K'' + -K\,H'K' - H(K')^2}{4H^2K} = 0 \qquad (1.11)$$

and

$$G^4_4 = 0 \qquad (1.12)$$

The solution of (1.11) is

$$H = \frac{C_1 (K')^2}{K} \qquad (1.13)$$

We thus find

$$ds^2 = -\,(dx^2 + dy^2) - C_1 \frac{(K')^2}{K} dt^2 + K\,dt^2 \qquad (1.14)$$

also satisfies $G^i_j = 0$.

However, it is not difficult to prove that for the metric (1.14), all components of the Riemann Christoffel Tensor vanish identically. Hence, the metric (1.14) is flat. It can also be shown that (1.14) is the only flat metric belonging to the general class (1.1).

From (1.10) we may seek the general isotropic line element by setting

$$\frac{C_1}{K^2} = \frac{C_2 (K')^2}{K^5} \qquad (1.15)$$

Then,

$$K' = C\,K^{3/2} \tag{1.16}$$

or

$$K = (Cx + D)^2 \tag{1.17}$$

Hence

$$ds^2 = -\,(Cx + D)^{-4}(dx^2 + dy^2 + dz^2\,)$$
$$+\,(Cx + D)^2\,dt^2 \tag{1.18}$$

is the isotropic line element.

APPENDIX VII

FIELD THEORY OF GRAVITATION*

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SUMMARY

A field theory of gravitation where the stress-energy tensor of gravity contributes to geometric curvatures is described. The field is locally Lorentz covariant, namely, in an infinitesimally small laboratory (not necessarily freely falling) the equations are formally the same as they would be in special relativity. The theory reproduces all known gravitational effects correctly and has experimental consequences differing from those of the Einsteinian theory.

1. PRELIMINARY CONSIDERATIONS

We propose to construct a field theory of gravitation where special relativity is satisfied as a correspondence limit. To make the essence of this correspondence explicit we introduce two postulates: a) Matter tensor $\sigma u_\mu u^\nu$ generates a gravitational field which is a local second order tensor, b) Local velocity of the gravitational field is equal to c. We express the content of these two statements as

$$\Box^2 \phi_\mu^\nu = 4\pi\sigma\, u_\mu u^\nu \tag{1.1}$$

$$\partial_\nu \phi_\mu^\nu = 0 \tag{1.2}$$

It is evident that the two postulates are generalizations of their special relativistic counterparts in the context of gravity. The first postulate extends one of the most important consequences of special relativity, namely, the matter density, σ*, becomes a tensor* $\sigma u_\mu u^\nu$*. The second postulate requires* $\partial_\nu \phi^\nu_\mu = 0$ *otherwise a* $\Lambda\partial^\nu\partial_\alpha\phi^\alpha_\mu$ *could be added to the left hand side of (1.1) which alters the vacuum velocity of propagation for some components. Locality further implies that the absolute values of the fields are unobservable and that only the differences* $\tilde{\phi}(x) \rightarrow \tilde{\phi}(x) - \tilde{\phi}(x')$ *enter into the observable phenomena. Note that (1.2) may further be supplemented by* $\partial^\mu\partial_\nu\phi^\nu_\mu = 0$ *because when one contracts (1.1) the additive terms must still be absent from the propagation equation of the trace,* $\phi^\mu_\mu = \phi$*. The gravitational field so conceived must, of course, have a stress-energy tensor* t^ν_μ *which must add to the matter tensor as*

$$T^\nu_\mu = \sigma u_\mu u^\nu + t^\nu_\mu/4\pi \qquad (1.3)$$

and the equations of motion of test particles must be obtained as

$$\sigma \frac{d^2 x_\mu}{ds^2} = \partial_\nu \,(t^\nu_\mu/4\pi) \qquad (1.4)$$

both of which are prerequisites of having a local field theory in correspondence with special relativity.

As can be seen these equations would be contradictory within the flat space-time of special relativity because the usual conservation law $\partial_\nu T^\nu_\mu = 0$ *would, on account of (1.2), lead to separate conservation of matter and field counterparts. This would mean that there can be no interaction between matter and field, hence (1.4) would be meaningless. This difficulty arises because in flat space* ∂_ν *commutes with* $\Box^2$*. Clearly in adopting the above equations we have implicitly assumed*

a more general space-time where we require

$$\partial_\nu \Box^2 \neq \Box^2 \partial_\nu \tag{1.5}$$

The latter is realizable in a curved geometry.

2. DEVELOPMENT OF THE THEORY

In the curved space-time the total stress-energy tensor will be identified as the divergence-free tensor[1]

$$R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R = 8\pi(\sigma u_\mu u^\nu + t_\mu^\nu/4\pi) \tag{2.1}$$

Upon contraction we shall write R as a scalar relation

$$R = -8\pi\sigma + 4L_f + 4\partial^\alpha J_\alpha \tag{2.2}$$

where $L_f = -t/2$ is the field Lagrangian and $\partial^\alpha J_\alpha$ a divergence. Our aim is to determine L_f and $\partial^\alpha J_\alpha$ by comparing (2.2) with the geometric expression of R, namely,

$$R = -\Box^2 \ln\sqrt{-g} - \frac{1}{2}\left\{{\alpha \atop \mu\nu}\right\}\partial_\alpha g^{\mu\nu} - (\sqrt{-g})^{-1}\partial^2_{\mu\nu}(\sqrt{-g}\,g^{\mu\nu}) \tag{2.3}$$

Contracting (1.1) we first obtain the equation of ϕ

$$\Box^2\phi = 4\pi\sigma \tag{2.4}$$

which implies that in (2.3) we must have

$$\sqrt{-g} = e^{2\phi} \tag{2.5}$$

Then the last term of (2.3) (which is the only one that can contain the Lorentz condition) and (1.2) imply that $\sqrt{-g}\,g^{\mu\nu}$ is independent of ϕ.

In view of (2.5) this requirement implies, in turn, that $g_{\mu\nu}$ is of the form

$$\tilde{g} = \tilde{\eta} \cdot e^{2(\phi - 2\tilde{\phi})} \tag{2.6}$$

where $\tilde{\phi}$ is the symmetric field tensor ϕ_{μ}^{ν}. By expanding this metric in first and second order we have

$$g_{\mu\nu} = \eta_{\mu\nu}(1 + 2\phi + 2\phi^2) - 4\eta_{\mu\lambda}\phi_{\nu}^{\lambda} - 8\phi\eta_{\mu\lambda}\phi_{\nu}^{\lambda} + 8\eta_{\alpha\beta}\phi_{\mu}^{\alpha}\phi_{\nu}^{\beta} \tag{2.7}$$

hence, to the required order, the last two terms of (2.3) are

$$-\frac{1}{2}\{{}^{\alpha}_{\mu\nu}\}\partial_{\alpha}g^{\mu\nu} = -4\partial^{\lambda}\phi_{\alpha}^{\beta}\,\partial_{\lambda}\phi_{\beta}^{\alpha} + 2\partial^{\lambda}\phi\partial_{\lambda}\phi + 8\partial_{\beta}\phi_{\alpha}^{\lambda}\,\partial^{\alpha}\phi_{\lambda}^{\beta} \tag{2.8}$$

$$-(\sqrt{-g})^{-1}\partial^{2}_{\mu\nu}(\sqrt{-g}\,g^{\mu\nu}) = -4\partial^{\mu}\partial_{\nu}\phi_{\mu}^{\nu} - 8\partial_{\beta}\phi_{\alpha}^{\lambda}\partial^{\alpha}\phi_{\lambda}^{\beta} \tag{2.8'}$$

We therefore find L_f and $\partial^{\alpha}J_{\alpha}$ as

$$L_f = -\partial^{\lambda}\phi_{\beta}^{\alpha}\,\partial_{\lambda}\phi_{\alpha}^{\beta} + \frac{1}{2}\,\partial^{\lambda}\phi\partial_{\lambda}\phi \tag{2.9}$$

$$\partial^{\alpha}J_{\alpha} = -\,\partial^{\mu}\partial_{\nu}\,\phi_{\mu}^{\nu} = 0 \tag{2.10}$$

Concerning the equations of motion (1.4) it is not difficult to see, from the form of (2.9), that the canonical stress-energy of L_f will yield the local force[4]

$$\sigma\,\frac{d^2x_{\mu}}{ds^2} = -2\sigma u_{\alpha}u^{\beta}\,\partial_{\mu}\phi_{\beta}^{\alpha} + \sigma\partial_{\mu}\phi \tag{2.11}$$

Upon dropping σ these equations are found to be the geodesic equations of the space given by (2.6), namely,

$$\frac{d^2x_{\mu}}{ds^2} = \frac{1}{2}\,\partial_{\mu}g_{\alpha\beta}u^{\alpha}u^{\beta} = \{{}^{\alpha}_{\mu\beta}\}\,u_{\alpha}u^{\beta} \tag{2.12}$$

Thus, under the condition that $\eta_{\mu\nu}$ *is an inertial space the presence of gravity leads to a curving of that space as in (2.6). Moreover, to the approximation here described, the gravitational equations of motion are the same as the geodesic equations of the curved space so generated.*

3. APPLICATIONS

In the usual solar system applications the source is static so that $\phi_\mu^\nu \to \phi_0^0$*. Since in this case* $\phi_0^0 = \phi$*,* $\partial^0\phi = 0$ *the problem is very simple. From (2.6) we have*

$$g_{00} = e^{-2\phi} \quad , \quad -g_{ii} = e^{2\phi} \qquad (3.1)$$

where $\phi = M/r$*. The form of* g_{00} *and* g_{ii} *show*[2] *that the red shift and light bending are correctly predicted. As for the perihelion advance we transform (3.1) into polar coordinates. Since in this case* ϕ *is a function of* r *only, the geodesic equations of motion (2.11) yield the constants of the motion,* $u_0 = k$*,* $u_\vartheta = q$*,* $u_\varphi = h$*. Choosing* $\theta = \pi/2$*,* $u_\theta = 0$ *one has*

$$u^0 = ke^{2\phi} \quad , \quad u^\varphi = h \frac{e^{-2\phi}}{r^2} \qquad (3.2)$$

where $k \simeq 1$*. The line element now is*

$$ds^2 = e^{-2\phi}dt^2 - e^{2\phi}(dr^2 + r^2 d\varphi^2) \qquad (3.3)$$

Upon introducing (3.4) into (3.5) and eliminating dt and ds one has

$$\left(\frac{dr}{d\varphi}\right)^2 + r^2 = \frac{r^4}{h^2} e^{2\phi}(e^{2\phi} - 1) \qquad (3.4)$$

Setting $u = 1/r$, and differentiating one finds

$$\ddot{u} + u = f(u) = \frac{M}{h^2} e^{2\phi}(2e^{2\phi} - 1) \tag{3.5}$$

which, as is well-known,[2] yields the observed perihelion advance, $\Delta\Omega = \pi df/du\Big|_{u=a} = 6\pi M^2/h^2$, $a = f(0) = M/h^2$. Likewise a calculation of the radar-echo delay with (3.1) and (3.2) gives the observed value (ℓ and ℓ' are the orbital radius of the planet and the earth, respectively).

$$\Delta t = 4M\left(\ln 4 \frac{\ell\ell'}{R^2} - \frac{\ell - \ell'}{\ell'}\right) \tag{3.6}$$

to within the experimental accuracy.

In the weak quasistatic limit where $\phi_0^0 \simeq \phi$ is the largest components (2.6) yields

$$g_{\mu\nu} \simeq \eta_{\mu\nu}(1 + 2\phi + 2\phi^2) - 4\eta_{\mu\lambda}\phi_\nu^\lambda \tag{3.7}$$

A comparison with the usual PPN formalism then shows that all the preferred-frame parameters are zero hence no preferred frame effect is predicted. The theory therefore reproduces correctly all the experimental results concerning earth tides and to the absence of certain effects which would otherwise exist in a preferred frame theory. Thus at first sight the predictionsof the new theory seem to be the same as those of Einstein's theory.[3]

There are, however, situations in which the present theory differs drastically from Einstein's theory. For example the vanishing covariant divergence of (2.1) yields

$$(\sqrt{-g})^{-1}\partial_\nu(\sqrt{-g}\,\sigma u_\mu u^\nu) - \sigma \left\{{\alpha \atop \mu\beta}\right\} u_\alpha u^\beta + D_\nu(t_\mu^\nu/4\pi) = 0 \tag{3.8}$$

Since, in first and second order, the last term is the same as in (1.4) one finds

$$\partial_\nu(\sqrt{-g}\,\sigma u_\mu u^\nu) = 0 \tag{3.9}$$

which expresses the conservation of total energy-momentum of matter in the presence of gravity. It is well-known that Einstein's theory yields $\partial_\nu(\sqrt{-g}\,\sigma u^\nu) = 0$ *which is the conservation of total rest-mass. Since in the limit of special relativity (3.9) reduces correctly to the conservation of energy-momentum of matter, we are able to accommodate the increase of rest-mass when two objects coalesce in a radiationless collision (the so-called mass-defect) and thereby preserve* $m = E/c^2$. *The argument is in fact valid both in the strong and in the weak field limits and allows* $e^+ + e^- \rightarrow 2\gamma$.[4]

In the extremely strong field limit, near Schwarzchild radius, the behavior of (3.1) is seen to be drastically different. In fact there is no singularity except at $r \rightarrow 0$ *hence no black hole behavior in the usual sense is predicted.*[4,5] *The collapsed object will here remain as naked except for* $r = 0$. *In the case of neutron stars with radii considerably larger than* $a = GM/c^2$ *the static behavior is not very different from the predictions of the conventional theory. As for the highly non-static cases the theory predicts gravitational radiation from exploding galaxies, which might be detectable.*[6] *In contrast the Einsteinian theory seems to be unable to accommodate gravitational radiation. In a recent calculation with the help of powerful symbolic manipulation techniques it is shown that the conventional linearized plane wave*

$$g_{11} = -1 + 4Q \quad , \quad g_{22} = -1 - 4Q \quad , \quad g_{12} = 4R \tag{3.10}$$

where R and Q are functions of t - z would lead to contradictions in second order. More specifically it is found that only three special

cases admit a solution a) $Q = R =$ constant (flat space), b) $R = iQ$ (metric is not real), c) $R \equiv 0$ (no polarization) $g_{11} = -e^{4(K+Q)}$, $g_{22} = -e^{4(K-Q)}$ where K is second order. Substituting into $R_{\mu\nu} = 0$ one finds $Q = C\cdot(t-z)^{\lambda}$. (Unacceptable at $z \to \infty$; Q prescribed).

In the new theory it is seen that $\phi_2^2 = -\phi_1^1 = Q(t-z)$, $\phi_1^2 = R(t-z)$ yields the metric $g_{11} = -\cosh 4\Omega + (Q/\Omega)\sinh 4\Omega$, $g_{22} = -\cosh 4\Omega - (Q/\Omega)\sinh 4\Omega$, $g_{12} = (R/\Omega)\sinh 4\Omega$, $g_{00} = -g_{33} = 1$ yields a solution to the field equations hence

$$\tilde{\phi} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & Q & R \\ \cdot & \cdot & R & -Q \end{pmatrix} \tag{3.11}$$

is a plane gravitational wave. Interestingly this leads to (3.10) as its first order approximation. Likewise it can be shown that in Einstein's theory a first order solution, which is necessary in view of a 200 km/sec motion of the solar system relative to the center of galaxy, becomes unsatisfactory in second order. Again the new theory is free from the difficulty.

4. POSSIBLE EXTENSIONS

The present theory can be completed with regard to mechanical forces such as pressure, and forces of electromagnetic and nuclear nature, by following a well-known procedure. We shall not repeat this process which is more or less straightforward. Instead, we shall present a more speculative possibility which resembles the Einsteinian idea of a unified field theory whereby various fields are introduced into the framework in analogy to ϕ_μ^ν. Let (2.6) be written as

$$\tilde{g} = \tilde{\eta}\cdot e^{2(\phi - 2\tilde{\phi}) + 2\tilde{\varepsilon}} \tag{4.1}$$

where $\tilde{\varepsilon}$ is an antisymmetric tensor. We define a tensor $f_{\mu\nu}$ such that $\varepsilon_{\mu\nu}$ is the world-line integral of $f_{\mu\nu}$

$$\varepsilon_{\mu\nu} = \frac{1}{2} \int f_{\mu\nu}\, ds \tag{4.2}$$

Computing R it can be shown that the Lorentz conditions are automatically satisfied

$$\partial^{\mu}\partial_{\nu} f^{\nu}_{\mu} = 0 \tag{4.3}$$

If we define aconserved quantity, j_{μ}

$$\partial_{\nu} f^{\nu}_{\mu} = -\, 4\pi j_{\mu} \tag{4.4}$$

$$\partial^{\mu} j_{\mu} = 0 \tag{4.5}$$

we have the first set of Maxwell's equations plus the conservation of charge. Then the antisymmetry of $f_{\mu\nu}$ yields the second set

$$\partial_{\nu} f_{\mu\sigma} + \partial_{\sigma} f_{\nu\mu} + \partial_{\mu} f_{\sigma\nu} = 0 \tag{4.6}$$

It can further be shown that, by a suitable definition of $\{{}^{\alpha}_{\mu\beta}\}$, the sign of the electric charge, the acceleration caused by t^{ν}_{μ} becomes

$$\sigma \frac{d^2 x_{\mu}}{ds^2} = \partial_{\nu}(t^{\nu}_{\mu}/4\pi) = f_{\mu\nu} j^{\nu} \tag{4.7}$$

This leads to a Lorentz force

$$m \frac{d^2 x_{\mu}}{ds^2} = e f_{\mu\nu} u^{\nu} \tag{4.8}$$

where, by integration, σ and ρ are replaced by the rest-mass and the charge. We can see that, in at least a formal sense, equations similar to those of the electrodynamics are obtainable from such a process as a local field theory. However, the t^{ν}_{μ} of the non-symmetric part does not reproduce the Maxwell stress-tensor exactly. We shall point out only that, by a suitable introduction of magnetic poles,

hopes are recently entertained that an SU(3) symmetry for heavy particles might be obtainable.[7] *This direction may conceivably be a remedy to the above di-ficulty with the Maxwell tensor. The problem is under further considerations.*

Less speculative is, perhaps, the possibility of a pure cosmical scalar, Φ, of Dicke-Brans type. Again writing

$$g_{\mu\nu} = \eta_{\mu\nu}\, e^{2(\phi - 2\tilde{\phi}) + \lambda\Phi} \tag{4.9}$$

where λ is a small admixture parameter, we can see that such a field can only yield a conformal extension of the present theory. Furthermore if $\lambda\Phi > 0.01\phi$ one has to renormalize g_{00} in crder not to conflict with the red-shift experiment. A pure scalar of this type to represent the whole gravitation would of course be out of the question because it would not produce any bending for light. For a small admixture $\lambda \simeq 0.05 \sim 0.08$ the bending of light and radar echo delay are correspondingly reduced and the theory can be tested by experiment. The essential point is that such a field is possible as a conformal extension of the present theo.y. An extension of a different kind, namely, the possibility of a rest-mass, μ, for gravity seems to be unrealistic on account of the very long range of the gravitational interaction.

5. COVARIANCE

In Section 2 we have seen that the Lorentz condition $\partial_\nu \phi_\mu^\nu = 0$ leads, in first order, to the expression, $\partial_\nu(\sqrt{-g}\, g^{\mu\nu}) = 0$, which is reminiscent of the so-called harmonicity conditions. Since the usual harmonicity conditions are non-covariant a question arises as to the nature of the coordinate invariance satisfied by the new theory. We shall now show that although they reduce to each other in the trivial

case of cartesian coordinates our conditions and the usual harmonicity concept have, in general, nothing to do with each other.

Let a curved space $h_{\mu\nu}(x)$ *be defined over all curvilinear coordinates* $\eta_{\mu\nu}(x)$ *of special relativity such that*

$$h_{\mu\nu}(x) = \eta_{\mu\lambda}(x)g_{\lambda\nu}(x) \qquad (5.1)$$

where $\eta_{\mu\nu} = (1,1,1,1)$, $\eta_{\mu\nu} = (1,\ r^2,\ r^2\sin^2\theta,\ 1)$ *etc. The* $g_{\mu\nu}(x)$ *are functions of space-time variables and they act as congruent coefficients to* $\eta_{\mu\nu}(x)$. *Upon constructing the d'Alembertian of* $h_{\mu\nu}(x)$ *one finds*

$$\Box_h^2 = g^{\mu\lambda}(\sqrt{\eta})^{-1}\partial_\nu(\sqrt{\eta}\ \eta^{\lambda\nu}\partial_\mu) + (\sqrt{-g})^{-1}\partial_\nu(\sqrt{-g}\ g^{\mu\nu})\partial_\mu \qquad (5.2)$$

Since the first term of this expression is a linear combination of special relativistic d'Alembertians the local propagation velocity will be c if the second term vanishes. We have[8]

$$\partial_\nu(\sqrt{-g}\ g^{\mu\nu}) = 0 \qquad (5.3)$$

These relations are not coordinate conditions. On the contrary they are the requirements that the local velocity of propagation be c as in special relativity. Subject to this statement, which is one of our principles, the theory is covariant, namely, it allows all curvilinear substitutions permissible by the local universality of c. Note further that the conditions (5.3) do not, in general, imply the harmonicity of the coordinates (for example in spherical coordinates corresponding to (3.1) $\Box^2 r = -\ 2e^{-2\phi}/r \neq 0$*). It would, therefore, be quite inappropriate to call (5.3) harmonicity conditions. We suggest that they be called the "propagation conditions" since they have to do with the local propagation velocity, c. It is clear that the theory here proposed does not violate the principle of covariance.*

The coordinates (2.6) are not preferred coordinates. They are special only in the sense that they allow a local special relativistic field theory interpretation of gravity and a local special relativistic philosophy of measurement.

6. EQUIVALENCE

It must not be supposed that (2.6) and (2.12) can only accommodate curved spaces and their fields of acceleration. We can show, with the help of (5.3), that the theory accommodates accelerated frames of reference in an otherwise flat space. Let $\eta_{\mu\nu} = \delta_{\mu\nu}$ *and let* $h_{\mu\nu}(x)$ *be still a flat space. Since* $R^{\sigma}_{\mu\nu\lambda} = 0$*, everywhere, we also have* $R = 0$*, everywhere. Thus in (2.2)* $\sigma = 0$, $L_f = 0$, $\partial^{\alpha} J_{\alpha} = 0$*. Letting the acceleration to be uniform in the z-direction (2.12) yields*

$$\phi^0_0 = -\gamma z + C \tag{6.1}$$

Then the propagation conditions (5.3) are satisfied if $\phi^3_3 = \phi^0_0 = \psi$*. This field indeed also satisfies the other conditions such as* $\sigma = 0$, $R = 0$ *etc., above, hence*

$$g_{00} = e^{-2\psi} \quad , \quad -g_{33} = e^{-2\psi} \quad , \quad g_{11} = g_{22} = -1 \tag{6.2}$$

is a flat-space solution consistent with (2.6). It may be conjectured that this solution represents a kinematically accelerated frame of reference in an otherwise flat space of special relativity.

Note that if the conjecture is valid, namely, if the kinematical acceleration does not curve the space, as gravity would, then the equivalence of gravity and inertia will not be exact but restricted to the Newtonian limit $v \to 0$ *alone. In fact the two solutions (3.1) and (6.2) yield, respectively, the geodesic accelerations*

$$-\ddot{x}_\nu = \partial_\nu\phi[1 + \frac{v^2}{c^2}], \quad -\ddot{x}_\nu = \partial_\nu\psi[1 - \frac{v_z^2}{c^2}] \qquad (6.3)$$

For v → 0, these accelerations are equal but for v → c they are different. The conjecture seems to be testable by an orbiting gyroscope placed in a spherical shell Assuming $\phi = \psi$ the gyroscope will have, relative to the shell, a differential acceleration

$$- \Delta\ddot{x}_\nu = \partial_\nu\phi \; \frac{\overline{v}_x^2 + \overline{v}_y^2 + 2\overline{v}_z^2}{c^2} \qquad (6.4)$$

The gyroscope could conceivably be replaced by some atoms with high rotational energy or with light waves. Since for a good gyroscope the factor is of the order of 10^{-12} it is also conceivable that a Dicke-Braginski type experiment might turn out to be feasible.

The outcome of such an experiment would be of great importance to see if the theory of gravitation has to incorporate inertia only in the limit of v → 0 or more generally for all velocities. In the latter case the flat space solutions of the kind (6.2) would be discarded from the theory by saying that under kinematic accelerations space-time becomes curved. On the other hand if the force given by (6.4) exists the kinematic accelerations are representable by coordinate transformations ($R^\rho_{\mu\nu\lambda} = 0$) whereas gravitational accelerations are not. In fact if (6.4) exists the equations of motion (1.4) will have to be restated. Thus if $\eta_{\mu\nu}(x)$ is flat but nevertheless non-inertial the geodesic equations of motion (2.12) will contain inertial forces coming from $\eta_{\mu\nu}(x)$ as well as the gravitational forces arising from $g_{\mu\nu}(\tilde{\phi})$. Thus we arrive at a more general conception of space

$$h_{\mu\nu} = (\eta(x) \; g(\tilde{\phi}))_{\mu\nu} = g_{\mu\nu}(x,\tilde{\phi}) \qquad (6.5)$$

where $\tilde{\phi} = 0$, $h_{\mu\nu} = g_{\mu\nu}(x)$ contains, in general, arbitrarily accelerated frames such as rotation and accommodates even intrinsicly curved spaces

such as the presence of a cosmological constant, Λ. The presence of gravity, $\tilde{\phi} \neq 0$, would be a further structure above and beyond these. It is, however, encouraging to see that, at least in the case of linear acceleration the formulae (2.6) and (2.12) can accommodate an inertial field $\psi_0^0 = \psi_3^3 = \psi$, $\Box^2\psi \equiv 0$ so that the linear acceleration can be conceived as a gauge transformation, over the presence of gravity, as it was always assumed it would be the case. Incidentally, the locality condition (5.3) can be viewed either as a first order statement or as in (2.8') valid in second order as well. The difference contributes to L_f a divergence, $\partial_\alpha J^\alpha = 2\partial_\alpha(\phi_\beta^\lambda \partial^\beta \phi_\lambda^\alpha)$, hence has no effect on the theory.

7. QUANTIZATION

The classical field theory of gravitation so constructed may be quantized by simply replacing the Lagrangian (2.9) with an operator expression

$$L_f = :\, - \partial^\lambda \phi_\alpha^\beta\, \partial_\lambda \phi_\beta^\alpha + \frac{1}{2}\, \partial^\lambda \phi\, \partial_\lambda \phi\, : \tag{7.1}$$

where semicolons imply normal ordering. By a well known procedure of introducing a state vector, $|\;\rangle$, a unitary transformation, U, etc., the commutation relations can locally be found in the usual manner because the theory is locally Lorentz invariant. In other words the problem of quantization of a gravitational field is here reduced to the quantization of a special relativistic tensor field. In fact an interesting economy in formulation is realized as follows: In the usual special relativistic case two independent constraints $\langle|\phi_\mu^\mu - \phi|\rangle = 0$ and $\langle|\partial_\nu \phi_\mu^\nu|\rangle = 0$ are imposed on the expectation values. First of these is necessary for the consistency of the Lagrangian procedure whereas the second is needed for the consistency of quantization. In the new theory the two turn out to be equivalent hence only one is

necessary to impose. The reason for this is that if the solution (2.6) is substituted into (2.1) the d'Alembert operator $\Box^2$ *is the general d'Alembertian whereas when we contract (2.1) the d'Alembertian must reduce to a scalar d'Alembertian* $\Box^2_s$ *if* ϕ^μ_μ *is to be interpreted as a scalar,* ϕ. *The substitution just mentioned yields*

$$\text{trace}\,(\Box^2\phi^\nu_\mu) - \Box^2_s\phi = 4\partial^\alpha\phi\partial_\nu\phi^\nu_\alpha \tag{7.2}$$

hence $\partial_\nu\phi^\nu_\alpha = 0$ *implies* $\phi^\mu_\mu = \phi$ = *scalar. In the process of quantization we shall therefore need only the condition*[8]

$$<|\partial_\nu\phi^\nu_\mu|> = 0 \tag{7.3}$$

The commutation relations are

$$\tilde{\phi}^*(x')\,\tilde{\phi}(x) - \tilde{\phi}(x)\,\tilde{\phi}^*(x') = i\tilde{D}(x-x') \tag{7.4}$$

$$\tilde{D}(x-x') = \frac{1}{(2\pi)^4}\int_C d^4k\,\tilde{\varepsilon}_k\,\frac{e^{ik(x-x')}}{-k^2} \tag{7.5}$$

which, by a suitable substitution, are found to be equivalent to a Gupta type of quantization.[9] *Both the treatment of the subsidiary conditions (7.3) and the derivation of the equations of motion follow the Gupta procedure with similar end results. The process of quantization being fairly standard the main improvement in the subject of quantized gravity is here due to the improvement of the background theory. Since there exists hardly any possibility of testing the quantized theory in the foreseeable future we shall emphasize below only some formal matters which seem to show that the present theory meshes quite naturally with the microscopic laws of quantum mechanics.*

By virtue of the local Lorentz invariance of the background theory the quantization is here a literal adaptation of theusual method of

quantization with the added feature that $g_{\mu\nu}(\tilde{\phi})$ is now one of the observables. The metric is therefore given by its expectation value

$$\bar{g}_{\mu\nu} = \langle|g_{\mu\nu}|\rangle = \langle|\tilde{\eta} \cdot e^{2(\phi - 2\tilde{\phi})}|\rangle \tag{7.6}$$

Conceptually this is a large departure from the Einsteinian viewpoint where $g_{\mu\nu}(x)$ is itself the gravitational field. The following comments might be of interest: a) Defining the vacuum by $\tilde{\phi}|\rangle = 0$ and assuming no quanta, real or virtual, are present anywhere the expectation value of (7.6) reduces to $\eta_{\mu\nu}$ alone. Hence a completely empty space has the property of global Lorentz structure, b) If we introduce a prescribed classical source $\sigma u_\mu u^\nu$ allow a large number of quanta (real and/or virtual) the expectation value of $g_{\mu\nu}$ approaches the solutions (time-dependent and/or static) solutions of the theory, c) Due to the behavior of g_{00} as a negative exponential $g_{00} \to e^{-2\phi}$ the theory does not seem to have the self energy difficulty although a renormalization might still be required as a redefinition of the rest-mass or of $g_{\mu\nu}$, d) In principle a massive gravitational field with a rest mass μ is possible to conceive and this would seem to obviate the necessity of dealing with an infrared divergence due to zero graviton rest-mass. It is, however, difficult to justify such an assumption physically as gravity seems to be such a long-range force that it applies at distances of intergalactic size, e) Since in the absence of a gravitational stress-energy tensor, t_μ^ν, the two formulae $D_\nu(\sigma u_\mu u^\nu) = 0$ and $D_\nu(\sigma u^\nu) = 0$ are equivalent the microscopic theory of gravity presented in this section may be conceived as a kind of spontaneous symmetry breaking, f) Quantized form agrees with the cross-section for gravitational radiation stated previously[6]. According to the theory gravitons act as normal spin-2 particles with vanishing rest mass and standard (N_2) properties of polarization[10].

References

1. H. Yilmaz, Phys. Rev., 111, 1417 (1958); Phys. Rev. Lettrs. 20, 1399 (1971); Ann. of Phys., 81, 179 (1973).

2. H. P. Robertson and T. W. Noonan, "Relativity and Cosmology", W. B. Sounders Company, p. 149, 296 (1968).

3. H. Yilmaz, Nuovo Cimento Lettrs., 5, 309 (1972); Physics Today, p. 15, March 1973; Ann. of Phys., ibid.

4. H. Yilmaz, Nuovo Cimento Lettrs., 7, 337 (1973); Ann. of Phys., ibid. We note that the rest-mass is defined as $m = \int \sqrt{-g}\sigma \, dV = \int \sqrt{-g} \, m \, \delta(x) \, dV$. Similarly the energy-momentum is $P_\mu = mu_\mu = \int \sqrt{-g}\sigma \, u_\mu dV = \int \sqrt{-g} \, m \, u_\mu \delta(x) \, dV$, the $P_o = mu_o$ component of which is mass-energy. The latter leads to $P_o = mu_o = mg_{oo} dt/ds \to mc^2 - mM/r + \frac{1}{2} mv^2$ which corresponds to the classical limit of the conservation of energy. If two coliding particles coalesce, the rest-mass of the composite particle is larger than the sum of the rest masses of the constituents as in special relativity.

5. R. E. Clapp, Phys. Rev., D7, 345 (1973).

6. H. Yilmaz (unpublished 1973); Nuovo Cimento, 10B, 79 (1972).

7. J. Schwinger, "Particles Fields and Sources", Addison-Wesley Publishing Company, p. 253, (1970).

8. H. Yilmaz, Ann. of Phys., Ibid.

9. S. N. Gupta, "Recent Developments in General Relativity" Pergamon Press, pp. 251-258 (1962).

10. D. M. Eardley, et. al. Phys. Rev., D8, 3308 (1973).

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